A Basic Course in the Theory of Interest and Derivatives Markets:
A Preparation for the Actuarial Exam FM/2

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In memory of my parents

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Preface

This manuscripts is designed for an introductory course in the theory of interest and annuity. This manuscript is suitablefor a junior level course in the mathematics of finance.
A calculator, such as TI BA II Plus, either the solar or battery version, will be useful in solving many of the problems in this book. A recommended resource link for the use of this calculator can be found at

http://www.scribd.com/doc/517593/TI-BA-II-PLUS-MANUAL.

The recommended approach for using this book is to read each section, work on the embedded examples, and then try the problems. Answer keys are provided so that you check your numerical answers against the correct ones.
Problems taken from previous exams will be indicated by the symbol ‡.
 If you find any errors, I would appreciate hearing from you: mfinan@atu.edu
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The Basics of Interest Theory

A component that is common to all financial transactions is the investment of money at interest. When a bank lends money to you, it charges rent for the money. When you lend money to a bank (also known as making a deposit in a savings account), the bank pays rent to you for the money. In either case, the rent is called "interest".

In Sections 1 through 14, we present the basic theory concerning the study of interest. Our goal here is to give a mathematical background for this area, and to develop the basic formulas which will be needed in the rest of the book.

1 The Meaning of Interest

To analyze financial transactions, a clear understanding of the concept of interest is required. Interest can be defined in a variety of contexts, such as the ones found in dictionaries and encyclopedias. In the most common context, **interest** is an amount charged to a borrower for the use of the lender's money over a period of time. For example, if you have borrowed $100 and you promised to pay back $105 after one year then the lender in this case is making a profit of $5, which is the fee for borrowing his money. Looking at this from the lender's perspective, the money the lender is investing is changing value with time due to the interest being added. For that reason, interest is sometimes referred to as the **time value of money.**
Interest problems generally involve four quantities: principal(s), investment period length(s), interest rate(s), amount value(s).
The money invested in financial transactions will be referred to as the **principal**, denoted by P. The amount it has grown to will be called the **amount value** and will be denoted by A. The difference $I = A - P$ is the **amount of interest** earned during the period of investment. Interest expressed as a percent of the principal will be referred to as an **interest rate.**
Interest takes into account the risk of default (risk that the borrower can't pay back the loan). The risk of default can be reduced if the borrowers promise to release an asset of theirs in the event of their default (the asset is called **collateral**).
The unit in which time of investment is measured is called the **measurement period**. The most measurement period is one year but may be longer or shorter (could be days, months, years, decades, etc.).

Example 1.1
Which of the following may fit the definition of interest?
(a) The amount I owe on my credit card.
(b) The amount of credit remaining on my credit card.
(c) The cost of borrowing money for some period of time.
(d) A fee charged on the money you've earned by the Federal government.

Solution.
The answer is (c) ■

Example 1.2
Let $A(t)$ denote the amount value of an investment at time t years.
(a) Write an expression giving the amount of interest earned from time t to time $t + s$ in terms of A only.
(b) Use (a) to find the annual interest rate, i.e., the interest rate from time t years to time $t + 1$ years.

Solution.
(a) The interest earned during the time t years and $t+s$ years is

$$A(t+s) - A(t).$$

(b) The annual interest rate is

$$\frac{A(t+1) - A(t)}{A(t)} \blacksquare$$

Example 1.3
You deposit $1,000 into a savings account. One year later, the account has accumulated to $1,050.
(a) What is the principal in this investment?
(b) What is the interest earned?
(c) What is the annual interest rate?

Solution.
(a) The principal is $1,000.
(b) The interest earned is $1,050 - $1,000 = $50.
(c) The annual interest rate is $\frac{50}{1000} = 5\%$ ■

Interest rates are most often computed on an annual basis, but they can be determined for non-annual time periods as well. For example, a bank offers you for your deposits an annual interest rate of 10% "compounded" semi-annually. What this means is that if you deposit $1000 now, then after six months, the bank will pay you $5\% \times 1000 = \$50$ so that your account balance is $1050. Six months later, your balance will be $5\% \times 1050 + 1050 = \$1102.50$. So in a period of one year you have earned $102.50. The annual interest rate is then 10.25% which is higher than the quoted 10% that pays interest semi-annually.
In the next several sections, various quantitative measures of interest are analyzed. Also, the most basic principles involved in the measurement of interest are discussed.

Practice Problems

Problem 1.1
You invest $3,200 in a savings account on January 1, 2004. On December 31, 2004, the account has accumulated to $3,294.08. What is the annual interest rate?

Problem 1.2
You borrow $12,000 from a bank. The loan is to be repaid in full in one year's time with a payment due of $12,780.
(a) What is the interest amount paid on the loan?
(b) What is the annual interest rate?

Problem 1.3
The current interest rate quoted by a bank on its savings accounts is 9% per year. You open an account with a deposit of $1,000. Assuming there are no transactions on the account such as depositing or withdrawing during one full year, what will be the amount value in the account at the end of the year?

Problem 1.4
The simplest example of interest is a loan agreement two children might make: "I will lend you a dollar, but every day you keep it, you owe me one more penny." Write down a formula expressing the amount value after t days.

Problem 1.5
When interest is calculated on the original principal only it is called **simple interest**. Accumulated interest from prior periods is not used in calculations for the following periods. In this case, the amount value A, the principal P, the period of investment t, and the annual interest rate i are related by the formula $A = P(1 + it)$. At what rate will $500 accumulate to $615 in 2.5 years?

Problem 1.6
Using the formula of the previous problem, in how many years will 500 accumulate to 630 if the annual interest rate is 7.8%?

Problem 1.7
Compounding is the process of adding accumulated interest back to the principal, so that interest is earned on interest from that moment on. In this case, we have the formula $A = P(1+i)^t$ and we call i a **yearly compound interest**. You can think of compound interest as a series of back-to-back simple interest contracts. The interest earned in each period is added to the principal of the previous period to become the principal for the next period.
You borrow $10,000 for three years at 5% annual interest compounded annually. What is the amount value at the end of three years?

Problem 1.8
Using compound interest formula, what principal does Andrew need to invest at 15% compounding annually so that he ends up with $10,000 at the end of five years?

Problem 1.9
Using compound interest formula, what annual interest rate would cause an investment of $5,000 to increase to $7,000 in 5 years?

Problem 1.10
Using compound interest formula, how long would it take for an investment of $15,000 to increase to $45,000 if the annual compound interest rate is 2%?

Problem 1.11
You have $10,000 to invest now and are being offered $22,500 after ten years as the return from the investment. The market rate is 10% compound interest. Ignoring complications such as the effect of taxation, the reliability of the company offering the contract, etc., do you accept the investment?

Problem 1.12
Suppose that annual interest rate changes from one year to the next. Let i_1 be the interest rate for the first year, i_2 the interest rate for the second year, $\cdots$, i_n the interest rate for the nth year. What will be the amount value of an investment of P at the end of the nth year?

Problem 1.13
Discounting is the process of finding the present value of an amount of cash at some future date. By the present value we mean the principal that must be invested now in order to achieve a desired accumulated value over a specified period of time. Find the present value of $100 in five years time if the annual compound interest is 12%.

Problem 1.14
Suppose you deposit $1000 into a savings account that pays annual interest rate of 0.4% compounded quarterly (see the discussion at the end of page 11.)
(a) What is the balance in the account at the end of year.
(b) What is the interest earned over the year period?
(c) What is the effective interest rate?

Problem 1.15
The process of finding the present value P of an amount A, due at the end of t years, is called **discounting** A. The difference $A - P$ is called the **discount on** A. Notice that the discount on A is also the interest on P. For example, if $1150 is the discounted value of $1250, due at the end of 7 months, the discount on the $1250 is $100. What is the interest on $1150 for the same period of time?

2 Accumulation and Amount Functions

Imagine a fund growing at interest. It would be very convenient to have a function representing the accumulated value, i.e. principal plus interest, of an invested principal at any time. Unless stated otherwise, we will assume that the change in the fund is due to interest only, that is, no deposits or withdrawals occur during the period of investment.
If t is the length of time, measured in years, for which the principal has been invested, then the amount of money at that time will be denoted by $A(t)$. This is called the **amount function.** Note that $A(0)$ is just the principal P.
Now, in order to compare various amount functions, it is convenient to define the function

$$a(t) = \frac{A(t)}{A(0)}.$$

This is called the **accumulation function.** It represents the accumulated value of a principal of 1 invested at time $t \geq 0$. Note that $A(t)$ is just a constant multiple of $a(t)$, namely $A(t) = A(0)a(t)$. That is, $A(t)$ is the accumulated value of an original investment of $A(0)$.

Example 2.1
Suppose that $A(t) = \alpha t^2 + 10\beta$. If X invested at time 0 accumulates to \$500 at time 4, and to \$1,000 at time 10, find the amount of the original investment, X.

Solution.
We have $A(0) = X = 10\beta; A(4) = 500 = 16\alpha + 10\beta$; and $A(10) = 1000 = 100\alpha + 10\beta$. Using the first equation in the second and third we obtain the following system of linear equations

$$\begin{aligned} 16\alpha + X =& 500 \\ 100\alpha + X =& 1000. \end{aligned}$$

Multiply the first equation by 100 and the second equation by 16 and subtract to obtain $1600\alpha + 100X - 1600\alpha - 16X = 50000 - 16000$ or $84X = 34000$. Hence, $X = \frac{34000}{84} = \$404.76$ ■

What functions are possible accumulation functions? Ideally, we expect $a(t)$ to represent the way in which money accumulates with the passage of time. Hence, accumulation functions are assumed to possess the following properties:
(P1) $a(0) = 1$.
(P2) $a(t)$ is increasing,i.e., if $t_1 < t_2$ then $a(t_1) \leq a(t_2)$. (A decreasing accumulation function implies a negative interest. For example, negative interest occurs when you start an investment with \$100 and at the end of the year your investment value drops to \$90. A constant accumulation function

implies zero interest.)
(P3) If interest accrues for non-integer values of t, i.e. for any fractional part of a year, then $a(t)$ is a continuous function. If interest does not accrue between interest payment dates then $a(t)$ possesses discontinuities. That is, the function $a(t)$ stays constant for a period of time, but will take a jump whenever the interest is added to the account, usually at the end of the period. The graph of such an $a(t)$ will be a step function.

Example 2.2
Show that $a(t) = t^2 + 2t + 1$, where $t \geq 0$ is a real number, satisfies the three properties of an accumulation function.

Solution.
(a) $a(0) = 0^2 + 2(0) + 1 = 1$.
(b) $a'(t) = 2t + 2 > 0$ for $t \geq 0$. Thus, $a(t)$ is increasing.
(c) $a(t)$ is continuous being a quadratic function ■

Example 2.3
Figure 2.1 shows graphs of different accumulation functions. Describe real-life situations where these functions can be encountered.

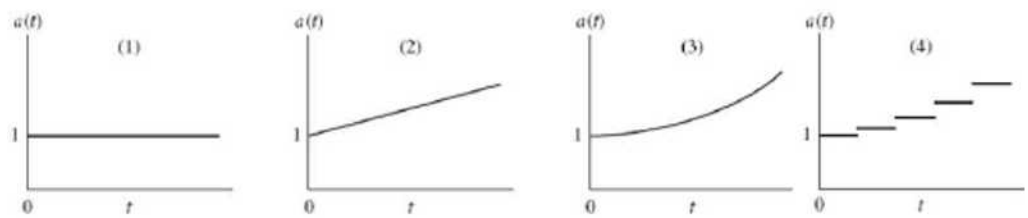


Figure 2.1

Solution.
(1) An investment that is not earning any interest.
(2) The accumulation function is linear. As we shall see in Section 4, this is referred to as "simple interest", where interest is calculated on the original principal only. Accumulated interest from prior periods is not used in calculations for the following periods.
(3) The accumulation function is exponential. As we shall see in Section 6, this is referred to as "compound interest", where the fund earns interest on the interest.
(4) The graph is a step function, whose graph is horizontal line segments of unit length (the period). A situation like this can arise whenever interest is paid out at fixed periods of time. If the amount of interest paid is constant per time period, the steps will all be of the same height. However, if the amount of interest increases as the accumulated value increases, then we would expect the steps to get larger and larger as time goes ■

Remark 2.1
Properties (P2) and (P3) clearly hold for the amount function $A(t)$. For example, since $A(t)$ is a positive multiple of $a(t)$ and $a(t)$ is increasing, we conclude that $A(t)$ is also increasing.

The amount function gives the accumulated value of k invested/deposited at time 0. Then it is natural to ask what if k is not deposited at time 0, say time $s > 0$, then what will the accumulated value be at time $t > s$? For example, \$100 is deposited into an account at time 2, how much does the \$100 grow by time 4?
Consider that a deposit of \$$k$ is made at time 0 such that the \$$k$ grows to \$100 at time 2 (the same as a deposit of \$100 made at time 2). Then $A(2) = ka(2) = 100$ so that $k = \frac{100}{a(2)}$. Hence, the accumulated value of \$$k$ at time 4 (which is the same as the accumulated value at time 4 of an investment of \$100 at time 2) is given by $A(4) = 100\frac{a(4)}{a(2)}$. This says that \$100 invested at time 2 grows to $100\frac{a(4)}{a(2)}$ at time 4.
In general, if \$$k$ is deposited at time s, then the accumulated value of \$$k$ at time $t > s$ is $k \times \frac{a(t)}{a(s)}$, and $\frac{a(t)}{a(s)}$ is called the **accumulation factor** or **growth factor.** In other words, the accumulation factor $\frac{a(t)}{a(s)}$ gives the dollar value at time t of \$1 deposited at time s.

Example 2.4
It is known that the accumulation function $a(t)$ is of the form $a(t) = b(1.1)^t + ct^2$, where b and c are constants to be determined.
(a) If \$100 invested at time $t = 0$ accumulates to \$170 at time $t = 3$, find the accumulated value at time $t = 12$ of \$100 invested at time $t = 1$.
(b) Show that $a(t)$ is increasing.

Solution.
(a) By (P1), we must have $a(0) = 1$. Thus, $b(1.1)^0 + c(0)^2 = 1$ and this implies that $b = 1$. On the other hand, we have $A(3) = 100a(3)$ which implies

$$170 = 100a(3) = 100[(1.1)^3 + c \cdot 3^2]$$

Solving for c we find $c = 0.041$. Hence,

$$a(t) = \frac{A(t)}{A(0)} = (1.1)^t + 0.041t^2.$$

It follows that $a(1) = 1.141$ and $a(12) = 9.042428377$.
Now, $100\frac{a(t)}{a(1)}$ is the accumulated value of \$100 investment from time $t = 1$ to $t > 1$. Hence,

$$100\frac{a(12)}{a(1)} = 100 \times \frac{9.042428377}{1.141} = 100(7.925002959) = 792.5002959$$

so \$100 at time $t = 1$ grows to \$792.50 at time $t = 12$.
(b) Since $a(t) = (1.1)^t + 0.041t^2$, we have $a'(t) = (1.1)^t \ln 1.1 + 0.082t > 0$ for $t \geq 0$. This shows that $a(t)$ is increasing for $t \geq 0$ ■

Now, let n be a positive integer. The n^{th}**period** of time is defined to be the period of time between $t = n - 1$ and $t = n$. More precisely, the period normally will consist of the time interval $n - 1 \leq t \leq n$.
We define the interest earned during the n^{th} period of time by

$$I_n = A(n) - A(n-1).$$

This is illustrated in Figure 2.2.

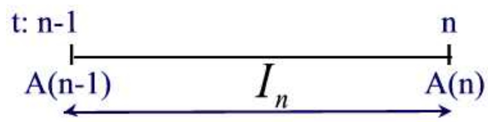


Figure 2.2

This says that interest earned during a period of time is the difference between the amount value at the end of the period and the amount value at the beginning of the period. It should be noted that I_n involves the effect of interest over an interval of time, whereas $A(n)$ is an amount at a specific point in time.
In general, the amount of interest earned on an original investment of \$$k$ between time s and t is

$$I_{[s,t]} = A(t) - A(s) = k(a(t) - a(s)).$$

Example 2.5
Consider the amount function $A(t) = t^2 + 2t + 1$. Find I_n in terms of n.

Solution.
We have $I_n = A(n) - A(n-1) = n^2 + 2n + 1 - (n-1)^2 - 2(n-1) - 1 = 2n + 1$ ■

Example 2.6
Show that $A(n) - A(0) = I_1 + I_2 + \cdots + I_n$. Interpret this result verbally.

Solution.
We have $A(n) - A(0) = [A(1) - A(0)] + [A(2) - A(1)] + \cdots + [A(n-1) - A(n-2)] + [A(n) - A(n-1)] = I_1 + I_2 + \cdots + I_n$. Hence, $A(n) = A(0) + (I_1 + I_2 + \cdots + I_n)$ so that $I_1 + I_2 + \cdots + I_n$ is the interest earned on the capital $A(0)$. That is, the interest earned over the concatenation of n periods is the

sum of the interest earned in each of the periods separately. ■

Note that for any $0 \le t < n$ we have $A(n) - A(t) = [A(n) - A(0)] - [A(t) - A(0)] = \sum_{j=1}^{n} I_j - \sum_{j=1}^{t} I_j = \sum_{j=t+1}^{n} I_j$. That is, the interest earned between time t and time n will be the total interest from time 0 to time n diminished by the total interest earned from time 0 to time t.

Example 2.7
Find the amount of interest earned between time t and time n, where $t < n$, if $I_r = r$ for some positive integer r.

Solution.
We have

$$\begin{aligned} A(n) - A(t) =& \sum_{i=t+1}^{n} I_i = \sum_{i=t+1}^{n} i \\ =& \sum_{i=1}^{n} i - \sum_{i=1}^{t} i \\ =& \frac{n(n+1)}{2} - \frac{t(t+1)}{2} = \frac{1}{2}(n^2 + n - t^2 - t) \end{aligned}$$

where we apply the following sum from calculus

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \blacksquare$$

Practice Problems

Problem 2.1
An investment of \$1,000 grows by a constant amount of \$250 each year for five years.
(a) What does the graph of $A(t)$ look like if interest is only paid at the end of each year?
(b) What does the graph of $A(t)$ look like if interest is paid continuously and the amount function grows linearly?

Problem 2.2
It is known that $a(t)$ is of the form $at^2 + b$. If \$100 invested at time 0 accumulates to \$172 at time 3, find the accumulated value at time 10 of \$100 invested at time 5.

Problem 2.3
Consider the amount function $A(t) = t^2 + 2t + 3$.
(a) Find the the corresponding accumulation function.
(b) Find I_n in terms of n.

Problem 2.4
Find the amount of interest earned between time t and time n, where $t < n$, if $I_r = 2^r$ for some positive integer r. Hint: Recall the following sum from Calculus: $\sum_{i=0}^{n} ar^i = a\frac{1-r^{n+1}}{1-r}, \ r \neq 1$.

Problem 2.5
\$100 is deposited at time $t = 0$ into an account whose accumulation function is $a(t) = 1 + 0.03\sqrt{t}$.
(a) Find the amount of interest generated at time 4, i.e., between $t = 0$ and $t = 4$.
(b) Find the amount of interest generated between time 1 and time 4.

Problem 2.6
Suppose that the accumulation function for an account is $a(t) = (1 + 0.5it)$. You invest \$500 in this account today. Find i if the account's value 12 years from now is \$1,250.

Problem 2.7
Suppose that $a(t) = 0.10t^2 + 1$. The only investment made is \$300 at time 1. Find the accumulated value of the investment at time 10.

Problem 2.8
Suppose $a(t) = at^2 + 10b$. If \$$X$ invested at time 0 accumulates to \$1,000 at time 10, and to \$2,000 at time 20, find the original amount of the investment X.

Problem 2.9
Show that the function $f(t) = 225 - (t - 10)^2$ cannot be used as an amount function for $t > 10$.

Problem 2.10
For the interval $0 \leq t \leq 10$, determine the accumulation function $a(t)$ that corresponds to $A(t) = 225 - (t-10)^2$.

Problem 2.11
Suppose that you invest \$4,000 at time 0 into an investment account with an accumulation function of $a(t) = \alpha t^2 + 4\beta$. At time 4, your investment has accumulated to \$5,000. Find the accumulated value of your investment at time 10.

Problem 2.12
Suppose that an accumulation function $a(t)$ is differentiable and satisfies the property

$$a(s+t) = a(s) + a(t) - a(0)$$

for all non-negative real numbers s and t.
(a) Using the definition of derivative as a limit of a difference quotient, show that $a'(t) = a'(0)$.
(b) Show that $a(t) = 1 + it$ where $i = a(1) - a(0) = a(1) - 1$.

Problem 2.13
Suppose that an accumulation function $a(t)$ is differentiable and satisfies the property

$$a(s+t) = a(s) \cdot a(t)$$

for all non-negative real numbers s and t.
(a) Using the definition of derivative as a limit of a difference quotient, show that $a'(t) = a'(0)a(t)$.
(b) Show that $a(t) = (1+i)^t$ where $i = a(1) - a(0) = a(1) - 1$.

Problem 2.14
Consider the accumulation functions $a_s(t) = 1 + it$ and $a_c(t) = (1+i)^t$ where $i > 0$. Show that for $0 < t < 1$ we have $a_c(t) \approx a_1(t)$. That is

$$(1+i)^t \approx 1 + it.$$

Hint: Expand $(1+i)^t$ as a power series.

Problem 2.15
Consider the amount function $A(t) = A(0)(1+i)^t$. Suppose that a deposit 1 at time $t = 0$ will increase to 2 in a years, 2 at time 0 will increase to 3 in b years, and 3 at time 0 will increase to 15 in c years. If 6 will increase to 10 in n years, find an expression for n in terms of a, b, and c.

Problem 2.16
For non-negative integer n, define

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)}.$$

Show that

$$(1 + i_n)^{-1} = \frac{A(n-1)}{A(n)}.$$

Problem 2.17
(a) For the accumulation function $a(t) = (1+i)^t$, show that $\frac{a'(t)}{a(t)} = \ln(1+i)$.
(b) For the accumulation function $a(t) = 1 + it$, show that $\frac{a'(t)}{a(t)} = \frac{i}{1+it}$.

Problem 2.18
Define

$$\delta_t = \frac{a'(t)}{a(t)}.$$

Show that

$$a(t) = e^{\int_0^t} \delta_r dr.$$

Hint: Notice that $\frac{d}{dr}(\ln a(r)) = \delta_r$.

Problem 2.19
Show that, for any amount function $A(t)$, we have

$$A(n) - A(0) = \int_0^n A(t)\delta_t dt.$$

Problem 2.20
You are given that $A(t) = at^2 + bt + c$, for $0 \leq t \leq 2$, and that $A(0) = 100, A(1) = 110$, and $A(2) = 136$. Determine $\delta_{\frac{1}{2}}$.

Problem 2.21
Show that if $\delta_t = \delta$ for all t then $i_n = \frac{a(n)-a(n-1)}{a(n-1)} = e^\delta - 1$. Letting $i = e^\delta - 1$, show that $a(t) = (1+i)^t$.

Problem 2.22
Suppose that $a(t) = 0.1t^2 + 1$. At time 0, \$1,000 is invested. An additional investment of \$X is made at time 6. If the total accumulated value of these two investments at time 8 is \$18,000, find X.

3 Effective Interest Rate (EIR)

Thus far, interest has been defined by

$$\text{Interest} = \text{Accumulated value} - \text{Principal}.$$

This definition is not very helpful in practical situations, since we are generally interested in comparing different financial situations to figure out the most profitable one. In this section we introduce the first measure of interest which is developed using the accumulation function. Such a measure is referred to as the **effective rate of interest**:
The effective rate of interest is the amount of money that one unit invested at the beginning of a period will earn during the period, with interest being paid at the end of the period.
If i is the effective rate of interest for the first time period then we can write

$$i = a(1) - a(0) = a(1) - 1.$$

where $a(t)$ is the accumulation function.

Remark 3.1
We assume that the principal remains constant during the period; that is, there is no contribution to the principal or no part of the principal is withdrawn during the period. Also, the effective rate of interest is a measure in which interest is paid at the end of the period compared to discount interest rate (to be discussed in Section 8) where interest is paid at the beginning of the period.

We can write i in terms of the amount function:

$$i = a(1) - a(0) = \frac{a(1) - a(0)}{a(0)} = \frac{A(1) - A(0)}{A(0)} = \frac{I_1}{A(0)}.$$

Thus, we have the following alternate definition:
The effective rate of interest for a period is the amount of interest earned in one period divided by the principal at the beginning of the period.

One can define the effective rate of interest for any period: The effective rate of interest in the n^{th} period is defined by

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)}$$

where $I_n = A(n) - A(n-1)$. Note that I_n represents the amount of growth of the function $A(t)$ in the n^{th} period whereas i_n is the rate of growth (based on the amount in the fund at the beginning of the period). Thus, the effective rate of interest i_n is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period.
Note that $i_1 = i = a(1) - 1$ and for any accumulation function, it must be true that $a(1) = 1 + i$.

Example 3.1
Assume that $A(t) = 100(1.1)^t$. Find i_5.

Solution.
We have

$$i_5 = \frac{A(5) - A(4)}{A(4)} = \frac{100(1.1)^5 - 100(1.1)^4}{100(1.1)^4} = 0.1 \blacksquare$$

Now, using the definition of i_n and solving for $A(n)$ we find

$$A(n) = A(n-1) + i_n A(n-1) = (1 + i_n)A(n-1).$$

Thus, the fund at the end of the n^{th} period is equal to the fund at the beginning of the period plus the interest earned during the period. Note that the last equation leads to

$$A(n) = (1 + i_1)(1 + i_2) \cdots (1 + i_n)A(0).$$

Example 3.2
If $A(4) = 1000$ and $i_n = 0.01n$, find $A(7)$.

Solution.
We have

$$\begin{aligned}
A(7) =&(1 + i_7)A(6)\\
=&(1 + i_7)(1 + i_6)A(5)\\
=&(1 + i_7)(1 + i_6)(1 + i_5)A(4) = (1.07)(1.06)(1.05)(1000) = 1,190.91 \blacksquare
\end{aligned}$$

Note that i_n can be expressed in terms of $a(t)$:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{A(0)a(n) - A(0)a(n-1)}{A(0)a(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)}.$$

Example 3.3
Suppose that $a(n) = 1 + in,\ n \geq 1$. Show that i_n is decreasing as a function of n.

Solution.
We have

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{[1 + in - (1 + i(n-1))]}{1 + i(n-1)} = \frac{i}{1 + i(n-1)}.$$

Since

$$i_{n+1} - i_n = \frac{i}{1 + in} - \frac{i}{1 + i(n-1)} = -\frac{i^2}{(1 + in)(1 + i(n-1))} < 0$$

we conclude that as n increases i_n decreases $\blacksquare$

Example 3.4
Show that if $i_n = i$ for all $n \geq 1$ then $a(n) = (1+i)^n$.

Solution.
We have

$$a(n) = \frac{A(n)}{A(0)} = (1+i_1)(1+i_2)\cdots(1+i_n) = (1+i)^n \blacksquare$$

Remark 3.2
In all of the above discussion the interest rate is associated with one complete period; this will be contrasted later with rates− called "nominal"− that are stated for one period, but need to be applied to fractional parts of the period. Most loans and financial products are stated with nominal rates such as a nominal rate that is compounded daily, or monthly, or semiannually, etc. To compare these loans, one compare their equivalent effective interest rates. Nominal rates will be discussed in more details in Section 9.

We pointed out in the previous section that a decreasing accumulated function leads to negative interest rate. We illustrate this in the next example.

Example 3.5
You buy a house for \$100,000. A year later you sell it for \$80,000. What is the effective rate of return on your investment?

Solution.
The effective rate of return is

$$i = \frac{80,000 - 100,000}{100,000} = -20\%$$

which indicates a 20% loss of the original value of the house $\blacksquare$

Practice Problems

Problem 3.1
Consider the accumulation function $a(t) = t^2 + t + 1$.
(a) Find the effective interest rate i.
(b) Find i_n.
(c) Show that i_n is decreasing.

Problem 3.2
If \$100 is deposited into an account, whose accumulation function is $a(t) = 1 + 0.03\sqrt{t}$, at time 0, find the effective rate for the first period(between time 0 and time 1) and second period (between time 1 and time 2).

Problem 3.3
Assume that $A(t) = 100 + 5t$.
(a) Find i_5.
(b) Find i_{10}.

Problem 3.4
Assume that $A(t) = 225 - (t - 10)^2,\ 0 \le t \le 10$. Find i_6.

Problem 3.5
An initial deposit of 500 accumulates to 520 at the end of one year and 550 at the end of the second year. Find i_1 and i_2.

Problem 3.6
A fund is earning 5% simple interest (See Problem 1.5). Calculate the effective interest rate in the 6th year.

Problem 3.7
Given $A(5) = 2500$ and $i = 0.05$.
(a) What is $A(7)$ assuming simple interest (See Problem 1.5)?
(b) What is $a(10)$?
Answer: (a) 2700 (b) 1.5

Problem 3.8
If $A(4) = 1200, A(n) = 1800$, and $i = 0.06$.
(a) What is $A(0)$ assuming simple interest?
(b) What is n?

Problem 3.9
John wants to have \$800. He may obtain it by promising to pay \$900 at the end of one year; or he may borrow \$1,000 and repay \$1,120 at the end of the year. If he can invest any balance over \$800 at 10% for the year, which should he choose?

Problem 3.10
Given $A(0) = \$1,500$ and $A(15) = \$2,700$. What is i assuming simple interest?

Problem 3.11
You invest \$1,000 now, at an annual simple interest rate of 6%. What is the effective rate of interest in the fifth year of your investment?

Problem 3.12
An investor purchases 1000 worth of units in a mutual fund whose units are valued at 4.00. The investment dealer takes a 9% "front-end load" from the gross payment. One year later the units have a value of 5.00 and the fund managers claim that the "fund's unit value has experienced a 25% growth in the past year." When units of the fund are sold by an investor, there is a redemption fee of 1.5% of the value of the units redeemed.
(a) If the investor sells all his units after one year, what is the effective annual rate of interest of his investment?
(b) Suppose instead that after one year the units are valued at 3.75. What is the return in this case?

Problem 3.13
Suppose $a(t) = 1.12^t - 0.05\sqrt{t}$.
(a) How much interest will be earned during the 5th year on an initial investment of \$12?
(b) What is the effective annual interest rate during the 5th year?

Problem 3.14
Assume that $A(t) = 100 + 5t$, where t is in years.
(a) Find the principal.
(b) How much is the investment worth after 5 years?
(c) How much is earned on this investment during the 5th year?

Problem 3.15
If \$64 grows to \$128 in four years at a constant effective annual interest rate, how much will \$10,000 grow to in three years at the same rate of interest?

Problem 3.16
Suppose that $i_n = 5\%$ for all $n \geq 1$. How long will it take an investment to triple in value?

Problem 3.17
You have \$1000 that you want to deposit in a savings account. Bank A computes the amount value of your investment using the amount function $A_1(t) = 1 + 0.049t$ whereas Bank B uses the amount function $A_2(t) = (1.4)^{12t}$. Where should you put your money?

Problem 3.18
Consider the amount function $A(t) = 12(1.01)^{4t}$.
(a) Find the principal.
(b) Find the effective annual interest rate.

Problem 3.19
Given $i_5 = 0.1$ and $A(4) = 146.41$. Find $A(5)$.

Problem 3.20
Given $i_5 = 0.1$ and $I_5 = 14.641$. Find $A(4)$.

Problem 3.21
Suppose that $i_n = 0.01n$ for $n \geq 1$. Show that $I_n = 0.01n(1.01)(1.02)\cdots[1 + 0.01(n-1)]A(0)$.

Problem 3.22
If $A(3) = 100$ and $i_n = 0.02n$, find $A(6)$.

4 Linear Accumulation Functions: Simple Interest

Accumulation functions of two common types of interest are discussed next. The accumulation function of "simple" interest is covered in this section and the accumulation function of "compound" interest is discussed in Section 6.

Consider an investment of 1 such that the interest earned in each period is constant and equals to i. Then, at the end of the first period, the accumulated value is $a(1) = 1 + i$, at the end of the second period it is $a(2) = 1 + 2i$ and at the end of the n^{th} period it is

$$a(n) = 1 + in, \quad n \geq 0.$$

Thus, the accumulation function is a linear function. The accruing of interest according to this function is called **simple interest.** Note that the effective rate of interest $i = a(1) - 1$ is also called the **simple interest rate**.
We next show that for a simple interest rate i, the effective interest rate i_n is decreasing . Indeed,

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{[1 + in - (1 + i(n-1))]}{1 + i(n-1)} = \frac{i}{1 + i(n-1)}, \; n \geq 1$$

and

$$i_{n+1} - i_n = \frac{i}{1 + in} - \frac{i}{1 + i(n-1)} = -\frac{i^2}{(1 + in)(1 + i(n-1))} < 0.$$

Thus, even though the rate of simple interest is constant over each period of time, the effective rate of interest per period is not constant− it is decreasing from each period to the next and converges to 0 in the long run. Because of this fact, simple interest is less favorable to the investor as the number of periods increases.

Example 4.1
A fund is earning 5% simple interest. Calculate the effective interest rate in the 6th year.

Solution.
The effective interest rate in the 6th year is i_6 which is given by

$$i_6 = \frac{i}{1 + i(n-1)} = \frac{0.05}{1 + 0.05(5)} = 4\% \blacksquare$$

Remark 4.1
For simple interest, the **absolute** amount of interest earned in each time interval, i.e., $I_n = a(n) - a(n-1)$ is constant whereas i_n is decreasing in value as n increases; in Section 6 we will see that under compound interest, it is the **relative** amount of interest that is constant, i.e. $i_n = \frac{a(n)-a(n-1)}{a(n-1)}$.

The accumulation function for simple interest has been defined for integral values of $n \geq 0$. In order for this function to have the graph shown in Figure 2.1(2), we need to extend $a(n)$ for nonintegral values of n. This is equivalent to crediting interest proportionally over any fraction of a period. If interest accrued only for completed periods with no credit for fractional periods, then the accumulation function becomes a step function as illustrated in Figure 2.1(4). Unless stated otherwise, it will be assumed that interest is allowed to accrue over fractional periods under simple interest.

In order to define $a(t)$ for real numbers $t \geq 0$ we will redefine the rate of simple interest in such a way that the previous definition is a consequence of this general assumption. The general assumption states the following:

Under simple interest, the interest earned by an initial investment of \$1 in all time periods of length $t + s$ is equal to the sum of the interest earned for periods of lengths t and s. Symbolically,

$$a(t+s) - a(0) = [a(t) - a(0)] + [a(s) - a(0)]$$

or

$$a(t+s) = a(t) + a(s) - a(0) \tag{4.1}$$

for all non-negative real numbers t and s.

Note that the definition assumes the rule is to hold for periods of any non-negative length, not just of integer length.

Are simple interest accumulation functions the only ones which preserve property (4.1)? Suppose that $a(t)$ is a differentiable function satisfying property (4.1). Then

$$\begin{aligned} a'(t) =& \lim_{s\to 0} \frac{a(t+s) - a(t)}{s} \\ =& \lim_{s\to 0} \frac{a(t) + a(s) - a(0) - a(t)}{s} \\ =& \lim_{s\to 0} \frac{a(s) - a(0)}{s} \\ =& a'(0), \quad \text{a constant} \end{aligned}$$

Thus the time derivative of $a(t)$ is shown to be constant. We know from elementary calculus that $a(t)$ must have the form

$$a(t) = a'(0)t + C$$

where C is a constant; and we can determine that constant by assigning to t the particular value 0, so that

$$C = a(0) = 1.$$

Thus,

$$a(t) = 1 + a'(0)t.$$

Letting $t = 1$ and defining $i_1 = i = a(1) - a(0)$ we can write

$$a(t) = 1 + it, \quad t \geq 0.$$

Consequently, simple interest accumulation functions are the only ones which preserve property (4.1).
It is important to notice that the above derivation does not depend on t being a nonnegative integer, and is valid for all nonnegative real numbers t.

Example 4.2
You invest \$100 at time 0, at an annual simple interest rate of 10%. Find the accumulated value after 6 months.

Solution.
The accumulated function for simple interest is a continuous function. Thus, $A(0.5) = 100[1 + 0.1(0.5)] = \$105$ ■

Remark 4.2
Simple interest is in general inconvenient for use by banks. For if such interest is paid by a bank, then at the end of each period, depositors will withdraw the interest earned and the original deposit and immediately redeposit the sum into a new account with a larger deposit. This leads to a higher interest earning for the next investment year. We illustrate this in the next example.

Example 4.3
Consider the following investments by John and Peter. John deposits \$100 into a savings account paying 6% simple interest for 2 years. Peter deposits \$100 now with the same bank and at the same simple interest rate. At the end of the year, he withdraws his balance and closes his account. He then reinvests the total money in a new savings account offering the same rate.
Who has the greater accumulated value at the end of two years?

Solution.
John's accumulated value at the end of two years is

$$100(1 + 0.06 \times 2) = \$112.$$

Peter's accumulated value at the end of two years is

$$100(1 + 0.06)^2 = \$112.36.$$

Thus, Peter has a greater accumulated value at the end of two years ■

Simple interest is very useful for approximating compound interest, a concept to be discussed in Section 6, for a short time period such as a fraction of a year. To be more specific, we will see that the accumulation function for compound interest i is given by the formula $a(t) = (1+i)^t$. Using the binomial theorem we can write the series expansion of $a(t)$ obtaining

$$(1+i)^t = 1 + it + \frac{t(t-1)}{2!}i^2 + \frac{t(t-1)(t-2)}{3!}i^3 + \cdots .$$

Thus, for $0 < t < 1$ we can write the approximation

$$(1+i)^t \approx 1 + it.$$

Example 4.4
\$10,000 is invested for four months at 12.6% compounded annually, that is $A(t) = 10000(1 + 0.126)^t$, where interest is computed using a quadratic to approximate an exact calculation. Find the accumulated value.

Solution.
We want to estimate $A(1/3) = 10000(1+0.126)^{\frac{1}{3}}$ using the first three terms of the series expansion of $(1+i)^t$ discussed above. That is,

$$\begin{aligned} A(1/3) =& 10000(1.126)^{\frac{1}{3}} \\ \approx& 10000\left(1 + \frac{1}{3} \times 0.126 + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{2!}(0.126)^2\right) = \$10,402.36 \blacksquare \end{aligned}$$

Remark 4.3
Under simple interest, what is the accumulated value at time $t > s$ of 1 deposited at time s? The SOA/CAS approach is different than the approach discussed in Section 2 right after Remark 2.1. According to the SOA/CAS, "simple interest" is generally understood to mean that the linear function starts all over again from the date of each deposit or withdrawal. We illustrate this approach in the next example ■

Example 4.5
Suppose you make a deposit of \$100 at time $t = 0$. A year later, you make a withdrawal of \$50. Assume annual simple interest rate of 10%, what is the accumulated value at time $t = 2$ years?

Solution.
With the SOA/CAS recommended approach for simple interest, the answer is

$$100(1 + 0.1 \times 2) - 50(1 + 0.1 \times 1) = \$65 \blacksquare$$

Practice Problems

Problem 4.1
You invest \$100 at time 0, at an annual simple interest rate of 9%.
(a) Find the accumulated value at the end of the fifth year.
(b) How much interest do you earn in the fifth year?

Problem 4.2
At what annual rate of simple interest will \$500 accumulate to \$615 in $2\frac{1}{2}$ years?

Problem 4.3
In how many years will \$500 accumulate to \$630 at 7.8% annual simple interest?

Problem 4.4
What principal will earn interest of 100 in 7 years at a simple interest rate of 6%?

Problem 4.5
What simple interest rate is necessary for \$10,000 to earn \$100 interest in 15 months?

Problem 4.6
At a certain rate of simple interest \$1,000 will accumulate to \$1110 after a certain period of time. Find the accumulated value of \$500 at a rate of simple interest three fourths as great over twice as long a period of time.

Problem 4.7
At time 0, you invest some money into an account earning 5.75% simple interest. How many years will it take to double your money?

Problem 4.8
You invest \$1,000 now, at an annual simple interest rate of 6%. What is the effective rate of interest in the fifth year of your investment?

Problem 4.9
Suppose that the accumulation function for an account is $a(t) = 1 + 3it$. At time 0, you invest \$100 in this account. If the value in the account at time 10 is \$420, what is i?

Problem 4.10
You have \$260 in a bank savings account that earns simple interest. You make no subsequent deposits in the account for the next four years, after which you plan to withdraw the entire account balance and buy the latest version of the iPod at a cost of \$299. Find the minimum rate of simple interest that the bank must offer so that you will be sure to have enough money to make the purchase in four years.

Problem 4.11
The total amount of a loan to which interest has been added is \$20,000. The term of the loan was four and one-half years. If money accumulated at simple interest at a rate of 6%, what was the amount of the loan?

Problem 4.12
If i_k is the rate of simple interest for period k, where $k = 1, 2, \cdots, n$, show that $a(n) - a(0) = i_1 + i_2 + \cdots + i_n$. Be aware that i_n is not the effective interest rate of the n^{th} period as defined in the section!

Problem 4.13
A fund is earning 5% simple interest. The amount in the fund at the end of the 5th year is \$10,000. Calculate the amount in the fund at the end of 7 years.

Problem 4.14
Simple interest of $i = 4\%$ is being credited to a fund. The accumulated value at $t = n - 1$ is $a(n-1)$. The accumulated value at $t = n$ is $a(n) = 1 + 0.04n$. Find n so that the accumulated value of investing $a(n-1)$ for one period with an effective interest rate of 2.5% is the same as $a(n)$.

Problem 4.15
A deposit is made on January 1, 2004. The investment earns 6% simple interest. Calculate the monthly effective interest rate for the month of December 2004.

Problem 4.16
Consider an investment with nonzero interest rate i. If i_5 is equal to i_{10}, show that interest is not computed using simple interest.

Problem 4.17
Smith has just filed his income tax return and is expecting to receive, in 60 days, a refund check of 1000.
(a) The tax service that helped him fill out his return offers to buy Smith's refund check for 850. What annual simple interest rate is implied?
(b) Smith negotiates and sells his refund check for 900. What annual simple interest rate does this correspond to?
(c) Smith deposits the 900 in an account which earns simple interest at annual rate 9%. How many days would it take from the time of his initial deposit of 900 for the account to reach 1000?

Problem 4.18
Let i be a simple interest rate and suppose that $i_6 = 0.04$. Calculate i.

Problem 4.19
A fund is earning 5% simple interest. If $i_n = 0.04$, calculate n.

Problem 4.20
Suppose that an account earns simple interest with annual interest rate of i. If an investment of k is made at time s years, what is the accumulated value at time $t > s$ years? Note that the answer is different from $k\frac{a(t)}{a(s)}$.

Problem 4.21
Suppose $A(5) = \$2,500$ and $i = 0.05$.
(a) What is $A(7)$ assuming simple interest?
(b) What is $a(10)$ assuming simple interest?

Problem 4.22
If $A(4) = \$1,200$ and $A(n) = \$1,800$,
(a) what is $A(0)$, assuming a simple interest of 6%?
(b) what is n if $i = 0.06$ assuming simple interest?

Problem 4.23
What is $A(15)$, if $A(0) = \$1,100$, simple interest is assumed and $i_n = 0.01n$?

5 Date Conventions Under Simple Interest

In the simple interest problems encountered thus far, the length of the investment has been an integral number of years. What happens if the time is given in days. In this section we discuss three techniques for counting the number of days in a period of investment or between two dates. In all three methods,

$$time = \frac{\text{\# of days between two dates}}{\text{\# of days in a year}}.$$

In what follows, it is assumed, unless stated otherwise, that in counting days interest is not credited for both the starting date and the ending date, but for only one of these dates.

Exact Simple Interest:
The "actual/actual" method is to use the exact number of days for the period of investment and to use 365 days in a nonleap year and 366 for a leap year (a year divisible by 4). Simple interest computed with this method is called **exact simple interest.** For this method, it is important to know the number of days in each month. In counting days between two dates, the last, but not the first, date is included.

Example 5.1
Suppose that \$2,500 is deposited on March 8 and withdrawn on October 3 of the same year, and that the interest rate is 5%. Find the amount of interest earned, if it is computed using exact simple interest. Assume non leap year.

Solution.
From March 8 (not included) to October 3 (included) there are $23+30+31+30+31+31+30+3 = 209$ days. Thus, the amount of interest earned using exact simple interest is $2500(0.05) \cdot \frac{209}{365} = \$71.58$ ■

Ordinary Simple Interest:
This method is also known as "30/360". The 30/360 day counting scheme was invented in the days before computers to make the computations easier. The premise is that for the purposes of computation, all months have 30 days, and all years have $12 \times 30 = 360$ days. Simple interest computed with this method is called **ordinary simple interest.**
The Public Securities Association (PSA) publishes the following rules for calculating the number of days between any two dates from $M_1/D_1/Y_1$ to $M_2/D_2/Y_2$:
• If D_1 (resp. D_2) is 31, change D_1 (resp. D_2)to 30.
• If M_1 (resp. M_2) is 2, and D_1 (resp. D_2) is 28 (in a non-leap year) or 29, then change D_1 (resp. D_2) to 30.
Then the number of days, N is:

$$N = 360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1).$$

For exampe, the number of days from February 25 to March 5 of the same year is 10 days. Like the exact simple interest, the ending date is counted and not the starting date.

Example 5.2
Jack borrows 1,000 from the bank on January 28, 1996 at a rate of 15% simple interest per year. How much does he owe on March 5, 1996? Use ordinary simple interest.

Solution.
The amount owed at time t is $A(t) = 1000(1 + 0.15t)$. Using ordinary simple interest with $Y_1 = Y_2 = 1996, M_1 = 1, M_2 = 3$, and $D_1 = 28$ and $D_2 = 5$ we find $t = \frac{37}{360}$ and the amount owed on March 5, 1996 is

$$1,000\left(1 + 0.15 \times \frac{37}{360}\right) = \$1,015.42 \blacksquare$$

Banker's Rule:
This method is also known as "actual/360". This method uses the exact number of days for the period of investment and that the calendar year has 360 days. Simple interest computed with this method is called **Banker's rule.** The number of days between two dates is found in the same way as for exact simple interest. In this method, we also count the last day but not the first day.

Example 5.3
Jack borrows 1,000 from the bank on January 1, 1996 at a rate of 15% simple interest per year. How much does he owe on January 17, 1996?Use Banker's rule.

Solution.
Jack owes

$$1000\left(1 + 0.15 \times \frac{16}{360}\right) = \$1,006.67 \blacksquare$$

Example 5.4
If an investment was made on the date the United States entered World War II, i.e., December 7, 1941, and was terminated at the end of the war on August 8, 1945, for how many days was the money invested under:
1. the "actual/actual" basis?
2. the "30/360" basis?

Solution.
1. From December 7, 1941(not included) to December 31, 1941 (included) there were 24 days. From January 1, 1942 to December 31, 1944 (including a leap year) there were $3(365) + 1 = 1096$ days.

From January 1, 1945 to August 8, 1945(included) the numbers of days is 31 + 28 + 31 + 30 + 31 + 30 + 31 + 8 = 220 days. The total number of days is 24 + 1096 + 220 = 1340.
2. We have $360(1945 - 1941) + 30(8 - 12) + (8 - 7) = 1321$ ■.

Remark 5.1
If the time is given in months, reduce it to a fraction of a year on the basis of 12 months to the year, without changing to days.

Example 5.5
A merchant is offered $50 discount for cash payment of a $1200 bill due in two months. If he pays cash, at what rate may he consider his money to be earning interest in the next two months?

Solution.
The merchant would pay now $1150 in place of $1200 in two months. To find the interest rate under which $1150 is the present value of $1200, due in two months, use the formula $I = Pit$ which by substitution becomes $50 = \frac{1150i}{6}$. Solving for i we find $i = 26.087\%$ ■

We end this section by pointing out that the methods discussed above do not only apply for simple interest rate problems but also to compound interest rate problems. Compound interest rates are introduced in the next section. Unless otherwise stated, in later sections we will assume always the actual/actual method is in use.

Practice Problems

Problem 5.1
Find the amount of interest that \$2,000 deposited on June 17 will earn, if the money is withdrawn on September 10 in the same year and if the simple rate of interest is 8% using (a) exact simple interest, (b) ordinary simple interest, and (c) Banker's rule. Assume non-leap year.

Problem 5.2
A sum of 10,000 is invested for the months of July and August at 6% simple interest. Find the amount of interest earned:
1. Assuming exact simple interest (Assume non-leap year).
2. Assuming ordinary simple interest.
3. Assuming the Banker's Rule.

Problem 5.3
Show that the Banker's Rule is always more favorable to the lender than is exact simple interest.

Problem 5.4
(a) Show that the Banker's Rule is usually more favorable to the lender than is ordinary simple interest.
(b) Give an example in (a) for which the opposite relationship holds.

Problem 5.5
Suppose that \$2,500 is deposited on March 8 and withdrawn on October 3 of the same year, and that the interest rate is 5%. Find the amount of interest earned, if it is computed using
(a) exact simple interest (Assume non-leap year),
(b) ordinary simple interest,
(c) the Banker's Rule.

Problem 5.6
The sum of \$ 5,000 is invested for the months of April, May, and June at 7% simple interest. Find the amount of interest earned
(a) assuming exact simple interest in a non-leap year;
(b) assuming exact simple interest in a leap year (with 366 days);
(c) assuming ordinary simple interest;
(d) assuming the Banker's Rule.

Problem 5.7
Fund A calculates interest using exact simple interest (actual/actual). Fund B calculates interest

using ordinary simple interest (30/360). Fund C calculates interest using the Banker's Rule (actual/360). All Funds earn 5% simple interest and have the same amount of money deposited on January 1, 2005.
Order the Funds based on the amount in the funds on March 1, 2005 from smallest to largest.

Problem 5.8
Suppose you lend \$60 to your sister on Sept 14, at an annual rate of simple interest of 10%, to be repaid on Dec 25. How much does she have to pay you back? Use actual/360 time measurement.

Problem 5.9
John borrows \$60 from Eddie. If he repays Eddie \$63 after 5 weeks, what simple interest has John paid? Use actual/actual for days counting. Assume non-leap year.

Problem 5.10
Henry invests 1000 on January 15 in an account earning simple interest at an annual effective rate of 10%. On November 25 of the same year, Henry withdraws all his money. How much money will Henry withdraw if the bank counts days:
(a) Using exact simple interest (ignoring February 29th)
(b) Using ordinary simple interest
(c) Using Banker's Rule

Problem 5.11
Let I_e denote the interest earned using the exact simple interest method and I_b the interest earned using the Banker's rule. Find the ratio $\frac{I_e}{I_b}$.

Problem 5.12
Suppose that the interest earned using the Banker's rule is \$40.58. What is the interest earned using the exact simple interest method?

Problem 5.13
Show that $I_b = I_e + \frac{I_e}{72}$ and $I_e = I_b - \frac{I_b}{73}$. Thus, the Banker's rule is more favorable to the investor than the actual/actual rule.

Problem 5.14
On January 1, 2000, you invested \$1,000. Your investment grows to \$1,400 by December 31, 2007. What was the exact simple interest rate at which you invested?

Problem 5.15
A 3% discount is offered for cash payment of a \$2500 bill, due at the end of 90 days. At what rate of simple interest earned over the 90 days is cash payment is made? Use the Banker's rule.

Problem 5.16
The terms of payment of a certain debt are: net cash in 90 days or 2% discount for cash in 30 days. At what rate of simple interest is earned if the discount is taken advantage of?

6 Exponential Accumulation Functions: Compound Interest

Simple interest has the property that the interest earned is not invested to earn additional interest. In contrast, **compound interest** has the property that the interest earned at the end of one period is automatically invested in the next period to earn additional interest.
We next find the accumulation function for compound interest. Starting with an investment of 1 and with compound interest rate i per period. At the end of the first period, the accumulated value is $1 + i$. At the end of the second period, the accumulated value is $(1 + i) + i(1 + i) = (1 + i)^2$. Continuing this way, we find that the accumulated value after t periods is given by the exponential function

$$a(t) = (1 + i)^t, \quad \text{for integral t} \geq 0.$$

Interest accruing according to this function is called **compound interest.** We call i the **rate of compound interest.**
From the above accumulation function, we can write

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = i.$$

Thus, the effective rate of interest for compound interest is constant.
The accumulation function for compound interest has been defined for nonnegative integers. In order to extend the domain to fractional periods we note that the compound interest accumulation function $a(t) = (1 + i)^t$ satisfies the property

$$(1 + i)^{t+s} = (1 + i)^t \cdot (1 + i)^s.$$

Thus, under compound interest we will require the accumulation function to satisfy the property

$$a(t + s) = a(t) \cdot a(s), \quad t, s \geq 0. \tag{6.1}$$

This formula says that under compound interest the amount of interest earned by an initial investment of 1 over $t+s$ periods is equal to the amount of interest earned if the investment is terminated at the end of t periods and the accumulated value at that point is immediately reinvested for an additional s periods.
Are compound interest accumulation functions the only ones which preserve property (6.1)? Assuming $a(t)$ is differentiable and satisfying the above property, then from the definition of the derivative

we have

$$\begin{aligned}a'(t) =& \lim_{s\to 0}\frac{a(t+s)-a(t)}{s}\\ =& \lim_{s\to 0}\frac{a(t)a(s)-a(t)}{s}\\ =& a(t)\lim_{s\to 0}\frac{a(s)-a(0)}{s}\\ =& a(t)a'(0)\end{aligned}$$

Hence,

$$\frac{a'(t)}{a(t)} = (\ln a(t))' = a'(0)$$

a constant for all $t \geq 0$. Hence,

$$\ln a(t) = a'(0)t + C$$

Since $a(0) = 1$, we obtain $C = 0$ so

$$\ln a(t) = a'(0)t.$$

Setting $t = 1$ and recalling that

$$a(1) - a(0) = i_1 = i$$

yields

$$a'(0) = \ln(1+i)$$

so

$$a(t) = (1+i)^t$$

and this is valid for all $t \geq 0$.
We thus see the graphical distinction between simple and compound interest: the graph of an accumulation function under simple interest is a straight line − a linear function; the graph of an accumulation function under compound interest is an exponential function.
We next present a comparison result between simple interest and compound interest.

Theorem 6.1
Let $0 < i < 1$. We have
(a) $(1+i)^t < 1 + it$ for $0 < t < 1$,
(b) $(1+i)^t = 1 + it$ for $t = 0$ or $t = 1$,
(c) $(1+i)^t > 1 + it$ for $t > 1$.

Proof.
(a) Suppose $0 < t < 1$. Let $f(i) = (1+i)^t - 1 - it$. Then $f(0) = 0$ and $f'(i) = t(1+i)^{t-1} - t$. Since $i > 0$, it follows that $1 + i > 1$, and since $t < 1$ we have $t - 1 < 0$. Hence, $(1+i)^{t-1} < 1$ and therefore $f'(i) = t(1+i)^{t-1} - t < 0$ for $0 < t < 1$. It follows that $f(i) < 0$ for $0 < t < 1$.
(b) Follows by substitution.
(c) Suppose $t > 1$. Let $g(i) = 1 + it - (1+i)^t$. Then $g(0) = 0$ and $g'(i) = t - t(1+i)^{t-1}$. Since $i > 0$, we have $1 + i > 1$. Since $t > 1$ we have $t - 1 > 0$. Hence $(1+i)^{t-1} > 1$ and therefore $g'(i) = t - t(1+i)^{t-1} < t - t = 0$. We conclude that $g(i) < 0$. This establishes part (c) ■

Thus, simple and compound interest produce the same result over one measurement period. Compound interest produces a larger return than simple interest for periods greater than 1 and smaller return for periods smaller than 1. See Figure 6.1.

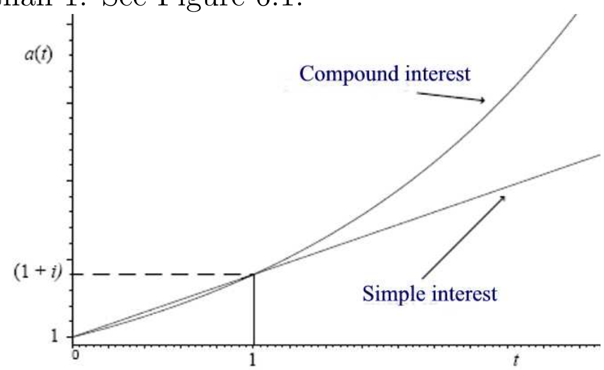


Figure 6.1

It is worth observing the following
(1) With simple interest, the absolute amount of growth is constant, that is, for a fixed s the difference $a(t+s) - a(t) = a(s) - 1$ does not depend on t.
(2) With compound interest, the relative rate of growth is constant, that is, for a fixed s the ratio $\frac{[a(t+s)-a(t)]}{a(t)} = a(s) - 1$ does not depend on t.

Example 6.1
It is known that \$600 invested for two years will earn \$264 in interest. Find the accumulated value of \$2,000 invested at the same rate of compound interest for three years.

Solution.
We are told that $600(1+i)^2 = 600 + 264 = 864$. Thus, $(1+i)^2 = 1.44$ and solving for i we find $i = 0.2$. Thus, the accumulated value of investing \$2,000 for three years at the rate $i = 20\%$ is $2,000(1+0.2)^3 = \$3,456$ ■

Example 6.2
At an annual compound interest rate of 5%, how long will it take you to triple your money? (Provide an answer in years, to three decimal places.)

Solution.
We must solve the equation $(1 + 0.05)^t = 3$. Thus, $t = \frac{\ln 3}{\ln 1.05} \approx 22.517$ ■

Example 6.3
At a certain rate of compound interest, 1 will increase to 2 in a years, 2 will increase to 3 in b years, and 3 will increase to 15 in c years. If 6 will increase to 10 in n years, find an expression for n in terms of a, b, and c.

Solution.
If the common rate is i, the hypotheses are that

$$\begin{aligned}
1(1+i)^a =& 2 \rightarrow \ln 2 = a \ln(1+i) \\
2(1+i)^b =& 3 \rightarrow \ln \frac{3}{2} = b \ln(1+i) \\
3(1+i)^c =& 15 \rightarrow \ln 5 = c \ln(1+i) \\
6(1+i)^n =& 10 \rightarrow \ln \frac{5}{3} = n \ln(1+i)
\end{aligned}$$

But

$$\ln \frac{5}{3} = \ln 5 - \ln 3 = \ln 5 - (\ln 2 + \ln 1.5).$$

Hence,

$$n \ln(1+i) = c \ln(1+i) - a \ln(1+i) - b \ln(1+i) = (c - a - b) \ln(1+i)$$

and this implies $n = c - a - b$ ■

We conclude this section by noting that the three counting days techniques discussed in Section 5 for simple interest applies as well for compound interest.

Example 6.4
Christina invests 1000 on April 1 in an account earning compound interest at an annual effective rate of 6%. On June 15 of the same year, Christina withdraws all her money. Assume non-leap year, how much money will Christina withdraw if the bank counts days:
(a) Using actual/actual method.
(b) Using 30/360 method.
(c) Using actual/360 method.

Solution.
(a) The number of days is 29 + 31 + 15 = 75. Thus, the amount of money withdrawn is

$$1000(1+0.06)^{\frac{75}{365}} = \$1,012.05.$$

(b) Using the 30/360, we find

$$1000(1+0.06)^{\frac{74}{360}} = \$1,012.05.$$

(c) Using the actual/360 method, we find

$$1000(1+0.06)^{\frac{75}{360}} = \$1,012.21 \blacksquare$$

Practice Problems

Problem 6.1
If \$4,000 is invested at an annual rate of 6.0% compounded annually, what will be the final value of the investment after 10 years?

Problem 6.2
Jack has deposited \$1,000 into a savings account. He wants to withdraw it when it has grown to \$2,000. If the interest rate is 4% annual interest compounded annually, how long will he have to wait?

Problem 6.3
At a certain rate of compound interest, \$250 deposited on July 1, 2005 has to accumulate to \$275 on January 1, 2006. Assuming the interest rate does not change and there are no subsequent deposits, find the account balance on January 1, 2008.

Problem 6.4
You want to triple your money in 25 years. What is the annual compound interest rate necessary to achieve this?

Problem 6.5
You invest some money in an account earning 6% annual compound interest. How long will it take to quadruple your account balance? (Express your answer in years to two decimal places.)

Problem 6.6
An amount of money is invested for one year at a rate of interest of 3% per quarter. Let $D(k)$ be the difference between the amount of interest earned on a compound interest basis, and on a simple interest basis for quarter k, where $k = 1, 2, 3, 4$. Find the ratio of $D(4)$ to $D(3)$.

Problem 6.7
Show that the ratio of the accumulated value of 1 invested at rate i for n periods, to the accumulated value of 1 invested at rate j for n periods, where $i > j$, is equal to the accumulated value of 1 invested for n periods at rate r. Find an expression for r as a function of i and j.

Problem 6.8
At a certain rate of compound interest an investment of \$1,000 will grow to \$1,500 at the end of 12 years. Determine its value at the end of 5 years.

Problem 6.9
At a certain rate of compound interest an investment of \$1,000 will grow to \$1,500 at the end of 12 years. Determine precisely when its value is exactly \$1,200.

Problem 6.10 ‡
Bruce and Robbie each open up new bank accounts at time 0. Bruce deposits 100 into his bank account, and Robbie deposits 50 into his. Each account earns the same annual effective interest rate.
The amount of interest earned in Bruce's account during the 11th year is equal to X. The amount of interest earned in Robbie's account during the 17th year is also equal to X.
Calculate X.

Problem 6.11
Given $A(5) = \$7,500$ and $A(11) = \$9,000$. What is $A(0)$ assuming compound interest?

Problem 6.12
If \$200 grows to \$500 over n years, what will \$700 grow to over $3n$ years? Assuming same compound annual interest rate.

Problem 6.13
Suppose that your sister repays you \$62 two months after she borrows \$60 from you. What effective annual rate of interest have you earned on the loan? Assume compound interest.

Problem 6.14
An investor puts 100 into Fund X and 100 into Fund Y. Fund Y earns compound interest at the annual rate of $j > 0$, and Fund X earns simple interest at the annual rate of $1.05j$. At the end of 2 years, the amount in Fund Y is equal to the amount in Fund X. Calculate the amount in Fund Y at the end of 5 years.

Problem 6.15
Fund A is invested at an effective annual interest rate of 3%. Fund B is invested at an effective annual interest rate of 2.5%. At the end of 20 years, the total in the two funds is 10,000. At the end of 31 years, the amount in Fund A is twice the amount in Fund B. Calculate the total in the two funds at the end of 10 years.

Problem 6.16
Carl puts 10,000 into a bank account that pays an annual effective interest rate of 4% for ten years. If a withdrawal is made during the first five and one-half years, a penalty of 5% of the withdrawal amount is made. Carl withdraws K at the end of each of years 4, 5, 6, 7. The balance in the account at the end of year 10 is 10,000. Calculate K.

Problem 6.17 ‡
Joe deposits 10 today and another 30 in five years into a fund paying simple interest of 11% per

year. Tina will make the same two deposits, but the 10 will be deposited n years from today and the 30 will be deposited $2n$ years from today. Tina's deposits earn an annual effective rate of 9.15% . At the end of 10 years, the accumulated amount of Tina's deposits equals the accumulated amount of Joe's deposits. Calculate n.

Problem 6.18
Complete the following table.

	Simple interest	Compound interest
a(t) =		
a(0) =		
a(1) =		
Period during which a(t) is greater		
$i_n =$		

Problem 6.19
On January 1, 2000, you invested \$1,000. Your investment grows to \$1,400 by December 31, 2007. What was the compound interest rate at which you invested?

Problem 6.20
A **certificate of deposit** or CD is a financial product commonly offered to consumers by banks and credit unions. A CD has a specific, fixed term (often three months, six months, or one to five years), and, usually, a fixed interest rate. It is intended that the CD be held until maturity, at which time the money may be withdrawn together with the accrued interest. However, a type a penalty is imposed for early withdrawal.
A two-year certificate of deposit pays an annual effective rate of 9%. The purchaser is offered two options for prepayment penalties in the event of early withdrawal:

A − a reduction in the rate of interest to 7%
B − loss of three months interest.

In order to assist the purchaser in deciding which option to select, compute the ratio of the proceeds under Option A to those under Option B if the certificate of deposit is surrendered:
(a) At the end of six months.
(b) At the end of 18 months.

Problem 6.21
You invest \$10,000 at time 0 into each of two accounts. Account A earns interest at an annual simple interest rate of 8%; Account B earns interest at an effective annual compound interest rate of 6%. What is the difference in the amount of interest earned during the 5th year in these two accounts?

Problem 6.22
On April 1, 2013, you will need \$10,000. Assuming a 6% annual compound effective rate of interest, what would you have to invest on October 1, 2008, in order for you to fulfill that need?

Problem 6.23
On January 1, 2009, you invest \$500 into an account earning an 8% annual compound effective interest rate. On January 1, 2011, you deposit an additional \$1,000 into the account. What is the accumulated value of your account on January 1, 2014?

Problem 6.24
Let i be a compound effective interest rate. Show that $I_n = iA(n-1)$.

Problem 6.25
Suppose that $A(1) = \$110, A(2) = \$121$, and $A(10) = \$259.37$.
(a) Find the effective rate of interest.
(b) Is the interest a simple interest or a compound interest?
(c) Find I_5.

Problem 6.26
If $A(5) = \$1,500$ and $A(n) = \$2,010.14$, what is n if $i = 0.05$ assuming compound interest?

Problem 6.27
How much interest is earned in the 9th year by \$200 invested today under compound interest with $i = 0.07$?

7 Present Value and Discount Functions

In this section we discuss the question of determining the today's or **present value** of some amount in the past or in the future. Unless stated otherwise, we assume in this section that we are in a compound interest situation, where $a(t) = (1+i)^t$.

Why would you be interested in present values? Suppose that in your personal financial life, you need to save or invest money that will grow to a specified amount in a specified number of years. For example, you may want to start saving now to buy a car for \$20,000 in two years, or to make a down payment of \$25,000 on a home in 3 years, or to pay a year's college expenses of \$30,000 in 8 years. All of these questions involve finding the present value (PV), i.e., the amount invested today that will grow to the desired amount in the desired time.

To start with, suppose we want to know what \$1 a period ago, invested at compound interest rate i per period, worths today. If X is the accumulated value then we must have $1(1+i) = X$. Thus, \$1 a period ago worths $1+i$ dollars now. We call $1+i$ the **accumulation factor**. Similarly, \$1 a period from now invested at the rate i worths $\nu = \frac{1}{1+i}$ today. We call ν the **discount factor** since it discounts the value of an investment at the end of a period to its value at the beginning of the period.

The above discussion can be generalized to more than one period. For example, \$1 invested t periods ago worths $(1+i)^t$ today and \$1 invested t periods from now is worth $\frac{1}{(1+i)^t}$ today. We call $\frac{1}{(1+i)^t}$ a **discount function.** It represents the amount that needs to be invested today at interest rate i per period to yield an amount of \$1 at the end of t time periods. This function can be expressed in terms of the accumulation function $a(t)$. Indeed, since $[a(t)]^{-1}a(t) = 1$, the discount function is $[a(t)]^{-1} = \frac{1}{(1+i)^t} = \nu^t$.

In a sense, accumulation and discounting are opposite processes. The term $(1+i)^t$ is said to be the **accumulated value** of \$1 at the end of t time periods. The term ν^t is said to be the **present value** or **discounted(back) value** of \$1 to be paid at the end of t periods.

Example 7.1
What is the present value of \$8,000 to be paid at the end of three years if the interest rate is 11% compounded annually?

Solution.
Let FV stands for the future value and PV for the present value. We want to find PV. We have $FV = PV(1+i)^3$ or $PV = FV(1+i)^{-3}$. Substituting into this equation we find $PV = 8000(1.11)^{-3} \approx \$5,849.53$ ■

Example 7.2
Show that the current value of a payment of 1 made $n > 1$ periods ago and a payment of 1 to be made n periods in the future is greater than 2, if $i > 0$.

Solution.
We have $(1+i)^n + (1+i)^{-n} = \left((1+i)^{\frac{n}{2}} - (1+i)^{-\frac{n}{2}}\right)^2 + 2 \geq 2$. Equality holds if and only if $(1+i)^{\frac{n}{2}} = (1+i)^{-\frac{n}{2}}$ which is equivalent to $(1+i)^n = 1$. This happens when either $i = 0$ or $n = 0$. Hence, $(1+i)^n + (1+i)^{-n} > 2$ since by assumption $n > 1$ and $i > 0$ ■

Example 7.3
Find an expression for the discount factor during the nth period from the date of investment, i.e. $(1+i_n)^{-1}$, in terms of the amount function.

Solution.
Recall that $i_n = \frac{A(n)-A(n-1)}{A(n-1)}$. Thus,

$$\frac{1}{1+i_n} = \frac{1}{1+\frac{A(n)-A(n-1)}{A(n-1)}} = \frac{A(n-1)}{A(n)} \blacksquare$$

Remark 7.1
One should observe that ν^t extend the definition of $a(t) = (1+i)^t$ to negative values of t. Thus, a graph of $a(t)$ is given in Figure 7.1.

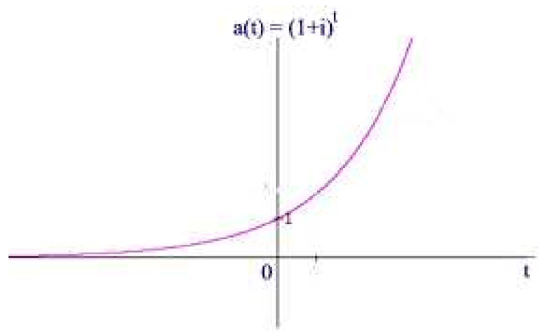


Figure 7.1

What happens to the calculation of present values if simple interest is assumed instead of compound interest? The accumulation function is now $a(t) = 1 + it$. Hence, the present value of 1, t years in the future, is

$$(a(t))^{-1} = \frac{1}{1+it}, \ t \geq 0.$$

Example 7.4
Find the present value of \$3,000 to be paid at the end of 5 years with a rate of simple interest of 7% per annum.

Solution.
The answer is $3000[a(5)]^{-1} = \frac{3000}{1+0.07(5)} \approx \$2,222.22$ ■

Example 7.5
Rework Example 7.4 using compound interest instead of simple interest.

Solution.
The answer is

$$\frac{3000}{1.07^5} = \$2,138.96$$ ■

Practice Problems

Problem 7.1
At an effective rate of interest of 8% per annum, the present value of \$100,000 due in X years is \$65,322. Determine X.

Problem 7.2
What deposit made today will provide for a payment of \$1,000 in 1 year and \$2,000 in 3 years, if the effective rate of interest is 7.5%?

Problem 7.3
The total amount of a loan to which interest has been added is \$20,000. The term of the loan was four and one-half years.
(a) If money accumulated at simple interest at a rate of 6%, what was the amount of the loan?
(b) If the annual rate of interest was 6% and interest was compounded annually, what was the amount of the loan?

Problem 7.4
It is known that an investment of \$500 will increase to \$4,000 at the end of 30 years. Find the sum of the present values of three payments of \$10,000 each which will occur at the end of 20, 40, and 60 years.

Problem 7.5
The sum of the present value of 1 paid at the end of n periods and 1 paid at the end of $2n$ periods is 1. Find $(1+i)^{2n}$.

Problem 7.6 ‡
Sally has two IRA's. IRA #1 earns interest at 8% effective annually and IRA #2 earns interest at 10% effective annually. She has not made any contributions since January 1, 1985, when the amount in IRA #1 was twice the amount in IRA #2. The sum of the two accounts on January 1, 1993 was \$75,000. Determine how much was in IRA #2 on January 1, 1985.

Problem 7.7 ‡
At an annual effective interest rate of $i, i > 0$, the following are all equal:
(i) the present value of 10,000 at the end of 6 years;
(ii) the sum of the present values of 6,000 at the end of year t and 56,000 at the end of year $2t$; and
(iii) 5,000 immediately.
Calculate the present value of a payment of 8,000 at the end of year $t+3$ using the same annual effective interest rate.

Problem 7.8
The Kelly family buys a new house for \$93,500 on May 1, 1996. How much was this house worth on May 1, 1992 if real estate prices have risen at a compound rate of 8% per year during that period?

Problem 7.9
Find the present (discounted) value of \$3,000 to be paid at the end of 5 years if the accumulation function is $a(t) = 1 + \frac{t^2}{25}$.

Problem 7.10
Suppose $a(t) = \alpha t^2 + 10\beta$. If \$X invested at time 0 accumulates to \$1,000 at time 10, and to \$2,000 at time 20, find the original amount of the investment X.

Problem 7.11
You want to buy a sailboat when you retire 5 years from now. The sailboat you wish to buy costs \$50,000 today and inflation is at 4% per year. What do you need to put into an account today to have enough funds to be able to purchase the boat 5 years from now if the account earns 7% per year?

Problem 7.12 ‡
A store is running a promotion during which customers have two options for payment. Option one is to pay 90% of the purchase price two months after the date of sale. Option two is to deduct X% off the purchase price and pay cash on the date of sale. A customer wishes to determine X such that he is indifferent between the two options when valuing them using an effective annual interest rate of 8%. Which of the following equations would the customer need to solve?
(A) $\left(\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right) = 0.90$
(B) $\left(1-\frac{X}{100}\right)\left(1+\frac{0.08}{6}\right) = 0.90$
(C) $\left(\frac{X}{100}\right)(1.08)^{\frac{1}{6}} = 0.90$
(D) $\left(\frac{X}{100}\right)\left(\frac{1.08}{1.06}\right) = 0.90$
(E) $\left(1-\frac{X}{100}\right)(1.08)^{\frac{1}{6}} = 0.90$

Problem 7.13
Derive a formula for i in terms of ν where $\nu = \frac{1}{1+i}$.

Problem 7.14
In this section the present value is assumed to be the value of an investment at time 0. We can determine the present value of an investment at any time n not necessarily at time 0.
A payment of \$10 is to be made at time 7 years.
(a) Determine the present value of this payment at time 4 years if the annual compound interest is 6%.
(b) Determine the present value of this payment at time 4 years if the annual simple interest is 6%.

Problem 7.15
A treasury bill or T−Bill with face value \$100 is a security which is exchangeable for \$100 on the maturity date. Suppose that a T−Bill with face value \$100 is issued on 08/09/2005 and matures on 03/09/2006 (there are 182 days between the dates).
Assuming simple interest, given that the annual rate of interest is 2.810%, find the price of the security.

Problem 7.16
If we are promised \$1,500 5 years from now, what is its present value today at: (a) 7% simple interest?
(b) 7% compound interest?

Problem 7.17
Show that the discount function of a simple interest rate i is larger than the discount function for a compound interest rate i for $t \geq 1$.

Problem 7.18
If $\frac{i}{1+i} = 0.06$, find i and ν.

Problem 7.19
What is the value 8 years from now of \$500 today if $\nu = 0.96$?

Problem 7.20 ‡
At an effective annual interest rate of $i, i > 0$, each of the following two sets of payments has present value K :
(i) A payment of 121 immediately and another payment of 121 at the end of one year.
(ii) A payment of 144 at the end of two years and another payment of 144 at the end of three years.
Calculate K.

8 Interest in Advance: Effective Rate of Discount

The effective rate of interest is defined as a measure of the interest paid at the end of the period. In this section, we introduce the **effective rate of discount**, denoted by d, which is a measure of interest where the interest is paid at the beginning of the period.
What do we mean by a statement such as "A loan of \$1,200 is made for one year at an effective rate of discount of 5%"? This means that the borrower will pay the interest of $1200 \times 0.05 = \$60$ (called the **amount of discount**)at the beginning of the year and repays \$1,200 at the end of the year. So basically, the lender is getting the interest in advance from the borrower.
In general, when $\$k$ is borrowed at a **discount rate** of d, the borrower will have to pay $\$kd$ in order to receive the use of $\$k$. Therefore, instead of the borrower having the use of $\$k$ at the beginning of a period he will only have the use of $\$(k - kd)$.
In our example above, we note that the effective rate of discount 5% is just the ratio

$$5\% = \frac{1200 - 1140}{1200}.$$

From this we can formulate the definition of effective rate of discount:
The **effective rate of discount** *is the ratio of the amount of discount during the period to the amount at the end of the period.*

Example 8.1
What is the difference between the following two situations?
(1) A loan of \$100 is made for one year at an effective rate of interest of 5%.
(2) A loan of \$100 is made for one year at an effective rate of discount of 5%.

Solution.
In both cases the fee for the use of the money is the same which is \$5. That is, the amount of discount is the same as the amount of interest. However, in the first case the interest is paid at the end of the period so the borrower was able to use the full \$100 for the year. He can for example invest this money at a higher rate of interest say 7% and make a profit of \$2 at the end of the transaction. In the second case, the interest is paid at the beginning of the period so the borrower had access to only \$95 for the year. So, if this amount is invested at 7% like the previous case then the borrower will make a profit of \$1.65. Also, note that the effective rate of interest is taken as a percentage of the balance at the beginning of the year whereas the effective rate of discount is taken as a percentage of the balance at the end of the year ■

Keep in mind the differences between the interest model and the discount model:
• Under the interest model, the payment for the use of the money is made at the end of the period,

based on the balance at the beginning of the period.

• Under the discount model, the payment for the use of the money is deducted at the beginning of the period from the final amount which will be present unchanged at the end of the period.

Even though an effective rate of interest is not the same as an effective rate of discount, there is a relationship between the two.

Assume that \$1 is invested for one year at an effective rate of discount d. Then the original principal is $\$(1-d)$. The effective rate of interest i for the year is defined to be the ratio of the amount of interest divided by the balance at the beginning of the year. That is,

$$i = \frac{d}{1-d} \tag{8.1}$$

Solving this last equation for d we find

$$d = \frac{i}{1+i}. \tag{8.2}$$

There is a verbal interpretation of this result: d is the ratio of the amount of interest that 1 will earn during the year to the balance at the end of the year.

Example 8.2

The amount of interest earned for one year when X is invested is \$108. The amount of discount earned when an investment grows to value X at the end of one year is \$100. Find X, i, and d.

Solution.

We have $iX = 108$, $\frac{i}{1+i}X = 100$. Thus, $\frac{108}{1+i} = 100$. Solving for i we find $i = 0.08 = 8\%$. Hence, $X = \frac{108}{0.08} = 1,350$ and $d = \frac{i}{1+i} = \frac{2}{27} \approx 7.41\%$ ■

Observe that several identities can be derived from Equations (8.1)-(8.2). For example, since $\nu = \frac{1}{1+i}$ we have

$$d = i\nu \tag{8.3}$$

that is, discounting i from the end of the period to the beginning of the period with the discount factor ν, we obtain d.

Next, we have

$$d = \frac{i}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} = 1 - \nu. \tag{8.4}$$

This is equivalent to $\nu = 1 - d$. Both sides of the equation represent the present value of 1 to be paid at the end of the period.

Now, from $i = \frac{d}{1-d}$ we have $i(1-d) = d$ or $i - id = d$. Adding $id - d$ to both sides we find

$$i - d = id \tag{8.5}$$

that is, the difference of interest in the two schemes is the same as the interest earned on amount d invested at the rate i for one period.

Effective rates of discount can be calculated over any particular measurement period:
"*The effective rate of discount d_n in the n^{th} period is defined to be the ratio of the amount of discount and the accumulated value at the end of the period*". That is

$$d_n = \frac{a(n) - a(n-1)}{a(n)}.$$

Since $A(n) = A(0)a(n)$, we can write

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{I_n}{A(n)}.$$

Example 8.3
If $a(t) = 1 + \frac{t^2}{25}$, find d_3.

Solution.
We have

$$d_3 = \frac{a(3) - a(2)}{a(3)} = \frac{1 + \frac{9}{25} - 1 - \frac{4}{25}}{1 + \frac{9}{25}} = \frac{5}{34} \approx 14.71\% \blacksquare$$

Analogous to the effective rate of interest i_n, d_n may vary from period to period. Recall that compound interest implies a constant rate of effective interest. A concept parallel to compound interest is compound discount. We say that d is a **compound discount** if it discounts \$1 according to the model shown in Figure 8.1, where t is the number of periods.

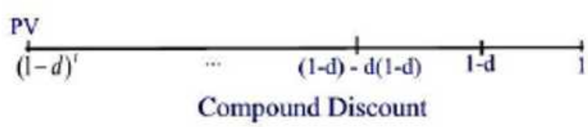


Figure 8.1

Let $a_c(t)$ be the accumulation function for compound discount $d > 0$ per period. From Figure 8.1, the original principal which will produce an accumulated value of 1 at the end of t periods is given by

$$[a_c(t)]^{-1} = (1-d)^t, t \geq 0.$$

Thus, the accumulation function for a compound discount d is

$$a_c(t) = \frac{1}{(1-d)^t}, t \geq 0.$$

Example 8.4
An investor would like to have $5,000 at the end of 20 years. The annual compound rate of discount is 5%. How much should the investor deposit today to reach that goal?

Solution.
The investor should set aside

$$5000(1-0.05)^{20} \approx \$1,792.43 \blacksquare$$

In the following theorem we prove that compound discount implies a constant rate of effective discount.

Theorem 8.1
Assuming compound discount $d > 0$ per period, we have $d_n = d$ for all $n \geq 1$.

Proof.
Compound interest and compound discount have equal accumulation functions. Indeed,

$$(1+i)^t = \left(1+\frac{d}{1-d}\right)^t = \frac{1}{(1-d)^t}, \quad \text{for all t} \geq 0.$$

Hence, we see that

$$d_n = \frac{a(n)-a(n-1)}{a(n)} = \frac{\frac{1}{(1-d)^n} - \frac{1}{(1-d)^{n-1}}}{\frac{1}{(1-d)^n}} = d \blacksquare$$

As for compound interest and compound discount, it is possible to define **simple discount** in a manner analogous to the definition of simple interest. We say that d is a **simple discount** if it discounts $1 according to the model shown in Figure 8.2.

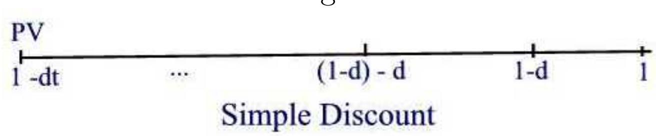


Figure 8.2

If $a_s(t)$ is the accumulation function for simple discount d, then from Figure 8.2 the original principal which will produce an accumulated value of 1 at the end of t periods is given by

$$[a_s(t)]^{-1} = 1 - dt, \quad 0 \leq t < \frac{1}{d}.$$

Note that, unlike the situation for simple interest, we must restrict the length of time over which we propose to apply simple discount. This is necessary to keep $[a_s(t)]^{-1} > 0$.

Thus, the accumulation function for simple discount d is

$$a_s(t) = \frac{1}{1-dt}, \ 0 \le t < \frac{1}{d}.$$

Example 8.5
Calculate the present value of a payment of 10,000 to be made in 17 years assuming a simple rate of discount of 3% per annum.

Solution.
The answer is $PV = 10000[1 - 0.03(17)] = \$4,900$ ■

It should be noted that formulas (15.3) - (8.5) assume effective rates of compound interest and discount (since $a(t) = a_c(t)$) and are not valid for simple rates of interest and discount unless the period of investment happens to be exactly one period. For example, if i and d are equivalent simple rate of interest and simple discount then $1 + it = \frac{1}{1-dt}$ for all $t \ge 0$. If (15.3) is satisfied then $(1 + it)(1 - dt) = 1$ is only valid for $t = 0$.

Example 8.6
If i and d are equivalent rates of simple interest and simple discount over t periods, show that $i - d = idt$.

Solution.
Since i and d are equivalent we must have $1+it = \frac{1}{1-dt}$ or $(1-dt)(1+it) = 1$. Thus, $1+ti-td-t^2id = 1$. This can be rearranged to obtain $ti - td = t^2id$. Dividing through by t, the result follows ■

Example 8.7
(a) Find d_5 if the rate of simple interest is 10%.
(b) Find d_5 if the rate of simple discount is 10%.

Solution.
(a) We are given that $i = 0.10$. Then

$$d_5 = \frac{a(5) - a(4)}{a(5)} = \frac{1 + 0.10(5) - (1 + 0.10(4))}{1 + 0.10(5)} = \frac{1}{15}.$$

(b) We are given $d = 0.10$. Then

$$d_5 = \frac{a_s(5) - a_s(4)}{a_s(5)} = \frac{\frac{1}{1-0.10(5)} - \frac{1}{1-0.10(4)}}{\frac{1}{1-0.10(5)}} = \frac{1}{6} \ ■$$

In the case of simple interest, i_n is a decreasing function of n. This is reversed for d_n with a simple discount rate.

Theorem 8.2
Assuming simple discount at a rate of discount $d > 0$, d_n is an increasing function of n for $0 < n-1 < \frac{1}{d}$.

Proof.
Under simple discount at a rate d of discount, we have $a_s(n) = \frac{1}{1-dn}$. Thus,

$$d_n = \frac{a_s(n) - a_s(n-1)}{a_s(n)} = \frac{\frac{1}{1-dn} - \frac{1}{1-d(n-1)}}{\frac{1}{1-dn}} = \frac{d}{1-dn+d} = \frac{d}{1+d(1-n)} > 0$$

since $0 < n-1 < \frac{1}{d}$. As n increases the denominator decreases so d_n increases ■

We next introduce the following definition: *Two rates of interest and/or discount are said to be* **equivalent** *if a given amount of principal invested over the same period of time at each of the rates produces the same accumulated value.*
We have seen in the process of proving Theorem 8.1 that an effective rate of compound interest i is equivalent to an effective rate of compound discount d since $a(t) = a_c(t)$.

Example 8.8
With i and d satisfying (8.1) - (8.5), show that

$$\frac{d^3}{(1-d)^2} = \frac{(i-d)^2}{1-\nu}.$$

Solution.
We have

$$\begin{aligned}\frac{d^3}{(1-d)^2} =& d\left(\frac{d}{1-d}\right)^2\\ =& di^2 = \frac{(id)^2}{d}\\ =& \frac{(i-d)^2}{1-\nu} \blacksquare\end{aligned}$$

Simple and compound discount produce the same result over one period. Over a longer period, simple discount produces a smaller present value than compound discount, while the opposite is true over a shorter period.

Theorem 8.3
Assuming $0 < d < 1$, we have
(a) $(1-d)^t < 1 - dt$ if $0 < t < 1$
(b) $(1-d)^t = 1 - dt$ if $t = 0$ or $t = 1$
(c) $(1-d)^t > 1 - dt$ if $t > 1$

Proof.
(a) Writing the power series expansion of $(1-d)^t$ we find

$$(1-d)^t = 1 - dt + \frac{1}{2}t(t-1)d^2 - \frac{1}{6}t(t-1)(t-2)d^3 + \cdots$$

If $t < 1$ then all terms after the second are negative. Hence, $(1-d)^t < 1 - dt$ for $0 < t < 1$.
(b) Straight forward by substitution.
(c) Suppose $t > 1$. Let $f(d) = 1 - dt - (1-d)^t$. Then $f(0) = 0$ and $f'(d) = -t + t(1-d)^{t-1}$. Since $t > 1$ we have $t - 1 > 0$. Since $0 < d < 1$, we have $1 - d < 1$. Thus, $(1-d)^{t-1} < 1$ and consequently $f'(d) < -t + t = 0$. Hence, $f(d) < 0$. This completes a proof of the theorem ■
Figure 8.3 compares the discount function under simple discount and compound discount.

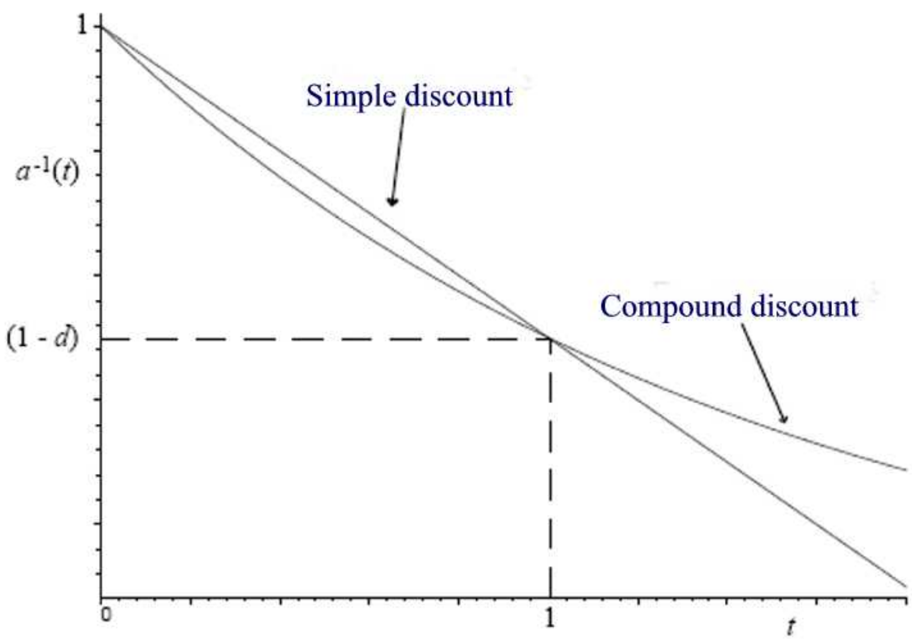


Figure 8.3

Practice Problems

Problem 8.1
The amount of interest earned on A for one year is \$336, while the equivalent amount of discount is \$300. Find A.

Problem 8.2
1000 is to be accumulated by January 1, 1995, at a compound discount of 9% per year.
(a) Find the present value on January 1, 1992.
(b) Find the value of i equivalent to d.

Problem 8.3
The amount of interest earned on A for one year is \$562.50, while the equivalent amount of discount is \$500. Find A.

Problem 8.4
Let i be a compound interest with equivalent compound discount d. Show the following
(a) $\frac{1}{d} - \frac{1}{i} = 1$.
(b) $d\left(1 + \frac{i}{2}\right) = i\left(1 - \frac{d}{2}\right)$.
(c) $i\sqrt{1-d} = d\sqrt{1+i}$.
Hint: Use $i - d = id$ for all three identities.

Problem 8.5
Calculate the present value of \$2000 payable in 10 years using an annual effective discount rate of 8%.

Problem 8.6
Calculate the accumulated value at the end of 3 years of 15,000 payable now assuming an interest rate equivalent to an annual discount rate of 8%.

Problem 8.7
An investor deposits \$1,000 today. The annual compound rate of discount is 6%. What is the accumulated value of the investment at the end of 10 years?

Problem 8.8
An investor would like to have \$10,000 at the end of 5 years. The annual simple rate of discount is 3%. How much should the investor deposit today to reach that goal?

Problem 8.9
An investor deposits \$5,000 today. The annual simple rate of discount is 5%. What is the accumulated value of the investment at the end of 7 months?

Problem 8.10
You are given that $\nu = 0.80$. Calculate d.

Problem 8.11
A fund earns interest at a rate equivalent to the rate of discount of d. Megan invested 10,000 in the fund. Eleven years later, Megan has 30,042. Calculate d.

Problem 8.12
In Account A, an investment of \$1,000 grows to \$1,700 in four years at an effective annual interest rate of x. In Account B, \$3,000 is invested for five years, at an effective rate of discount $d = x$. What is the ending balance in Account B?

Problem 8.13 ‡
Bruce and Robbie each open up new bank accounts at time 0 . Bruce deposits 100 into his bank account, and Robbie deposits 50 into his. Each account earns an annual effective discount rate of d. The amount of interest earned in Bruce's account during the 11th year is equal to X. The amount of interest earned in Robbie's account during the 17th year is also equal to X. Calculate X.

Problem 8.14
Suppose that $a(t) = 1.12^t$. What is the effective rate of discount in the 7th year?

Problem 8.15
A fund accumulates based on the following formula: $a(t) = \left(1 + \frac{t}{10}\right) \cdot 1.2^t$. Find d_3.

Problem 8.16
The effective rate of discount is 5%. Linda will receive \$500 one year from today, \$1,000 two years from today, and \$2,000 five years from today. Find the combined present value of her three future cash flows.

Problem 8.17
A business permits its customers to pay with a credit card or to receive a percentage discount of r for paying cash.
For credit card purchases, the business receives 97% of the purchase price one-half month later.
At an annual effective rate of discount of 22%, the two payments are equivalent. Find r.

Problem 8.18
A deposit of X is made into a fund which pays an annual effective interest rate of 6% for 10 years. At the same time , $\frac{X}{2}$ is deposited into another fund which pays an annual effective rate of discount of d for 10 years.
The amounts of interest earned over the 10 years are equal for both funds. Calculate d.

Problem 8.19
A signs a one-year note for \$1000 and receives \$920 from the bank. At the end of six months, A makes a payment of \$288. Assuming simple discount, to what amount does this reduce the face amount of the note?

Problem 8.20
A bank offers a 272−day discounted loan at a simple discount rate of 12%.
(a) How much money would a borrower receive if she asked for a \$5000 loan?
(b) What size loan should the borrower ask for in order to actually receive \$5000?
(c) What is the equivalent simple interest rate that is being charged on the loan?

Problem 8.21
A discounted loan of \$3000 at a simple discount rate of 6.5% is offered to Mr. Jones. If the actual amount of money that Mr. Jones receives is \$2869.11, when is the \$3000 due to be paid back?

Problem 8.22
Show that $d_n < i_n$.

Problem 8.23
At time $t = 0$, Paul deposits \$3500 into a fund crediting interest with an annual discount factor of 0.96. Find the fund value at time 2.5.

Problem 8.24
Method A assumes simple interest over final fractional periods, while Method B assumes simple discount over final fractional periods. The annual effective rate of interest is 20%. Find the ratio of the present value of a payment to be made in 1.5 years computed under method A to that computed under Method B.

9 Nominal Rates of Interest and Discount

When we speak of either the "effective" rate of interest/discount we mean interest is paid once per measurement period, either at the end of the period(in the case of interest rate) or at the beginning of the period(in the case of discount rate). In this section, we consider situations where interest is paid more than once per measurement period. Rates of interest and discount in these cases are called **nominal**. In this section we define nominal rates of interest and discount and determine relationships between nominal rates and effective rates as well as relationships between nominal rates of interest and discount. Compound interest or discount will always be assumed, unless specified otherwise.

When interest is paid (i.e., reinvested) more frequently than once per period, we say it is "payable" ("convertible", "compounded") each fraction of a period, and this fractional period is called the **interest conversion period.**

A **nominal rate of interest** $i^{(m)}$ payable m times per period, where m is a positive integer, represents m times the effective rate of compound interest used for each of the mth of a period. In this case, $\frac{i^{(m)}}{m}$ is the effective rate of interest for each mth of a period. Thus, for a nominal rate of 12% compounded monthly, the effective rate of interest per month is 1% since there are twelve months in a year.

Suppose that 1 is invested at a nominal rate $i^{(m)}$ compounded m times per measurement period. That is, the period is partitioned into m equal fractions of a period. At the end of the first fraction of the period the accumulated value is $1 + \frac{i^{(m)}}{m}$. At the end of the second fraction of the period the accumulated value is $\left(1 + \frac{i^{(m)}}{m}\right)^2$. Continuing, we find that the accumulated value at the end of the mth fraction of a period, which is the same as the end of one period, is $\left(1 + \frac{i^{(m)}}{m}\right)^m$ and at the end of t years the accumulated value is

$$a(t) = \left(1 + \frac{i^{(m)}}{m}\right)^{mt}.$$

Figure 9.1 illustrates accumulation at a nominal rate of interest for one measurement period.

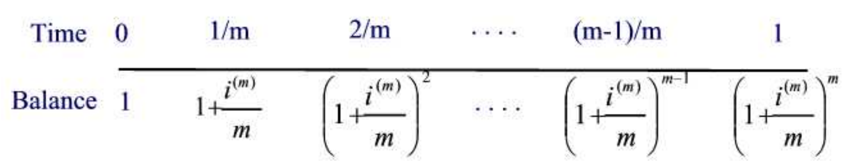


Figure 9.1

Example 9.1
Find the accumulated value of 1,000 after three years at a rate of interest of 24% per year convertible monthly.

Solution.
The accumulated value is $1,000\left(1+\frac{0.24}{12}\right)^{12\times 3}=2,039.89$ ■

Example 9.2
Find the accumulated value of \$3,000 to be paid at the end of 8 years with a rate of compound interest of 5%
(a) per annum;
(b) convertible quarterly;
(c) convertible monthly.

Solution.
(a) The accumulated value is $3,000\left(1+\frac{0.05}{1}\right)^{8}\approx \$4,432.37$.
(b) The accumulated value is $3,000\left(1+\frac{0.05}{4}\right)^{8\times 4}\approx \$4,464.39$.
(c) The accumulated value is $3,000\left(1+\frac{0.05}{12}\right)^{8\times 12}\approx \$4,471.76$ ■

Next we describe the relationship between effective and nominal rates. If i denotes the effective rate of interest per one measurement period equivalent to $i^{(m)}$ then we can write

$$1+i=\left(1+\frac{i^{(m)}}{m}\right)^m$$

since each side represents the accumulated value of a principal of 1 invested for one year. Rearranging we have

$$i=\left(1+\frac{i^{(m)}}{m}\right)^m-1$$

and

$$i^{(m)}=m[(1+i)^{\frac{1}{m}}-1].$$

For any $t\geq 0$ we have

$$(1+i)^t=\left(1+\frac{i^{(m)}}{m}\right)^{mt}.$$

Example 9.3
Given the nominal interest rate of 12%, compounded monthly. Find the equivalent effective annual interest rate.

Solution.
The answer is

$$i = \left(1 + \frac{0.12}{12}\right)^{12} - 1 = 12.7\% \blacksquare$$

Example 9.4
(a) Find the annual effective interest rate i which is equivalent to a rate of compound interest of 8% convertible quarterly.
(b) Find the compound interest rate $i^{(2)}$ which is equivalent to an annual effective interest rate of 8%.
(c) Find the compound interest rate $i^{(4)}$ which is equivalent to a rate of compound interest of 8% payable semi-annually.

Solution.
(a) We have

$$1 + i = \left(1 + \frac{0.08}{4}\right)^4 \Rightarrow i = \left(1 + \frac{0.08}{4}\right)^4 - 1 \approx 0.08243216$$

(b) We have

$$1 + 0.08 = \left(1 + \frac{i^{(2)}}{2}\right)^2 \Rightarrow i^{(2)} = 2[(1.08)^{\frac{1}{2}} - 1] \approx 0.07846.$$

(c) We have

$$\left(1 + \frac{i^{(4)}}{4}\right)^4 = \left(1 + \frac{i^{(2)}}{2}\right)^2 \Rightarrow i^{(4)} = 4[(1.04)^{\frac{1}{2}} - 1] \approx 0.0792 \blacksquare$$

In the same way that we defined a nominal rate of interest, we could also define a nominal rate of discount, $d^{(m)}$, as meaning an effective rate of discount of $\frac{d^{(m)}}{m}$ for each of the mth of a period with interest paid at the beginning of a mth of a period.
Figure 9.2 illustrates discounting at a nominal rate of discount for one measurement period.

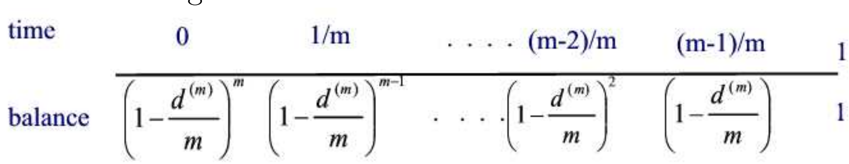


Figure 9.2

The accumulation function with the nominal rate of discount $d^{(m)}$ is

$$a(t) = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}, \quad t \geq 0.$$

Example 9.5
Find the present value of $8,000 to be paid at the end of 5 years at a an annual rate of compound interest of 7%
(a) convertible semiannually.
(b) payable in advance and convertible semiannually.

Solution.
(a) The answer is
$$\frac{8,000}{(1+\frac{0.07}{2})^{5\times 2}} \approx \$5,671.35$$
(b) The answer is
$$8000\left(1-\frac{0.07}{2}\right)^{5\times 2} \approx \$5,602.26 \blacksquare$$
If d is the effective discount rate equivalent to $d^{(m)}$ then
$$1-d=\left(1-\frac{d^{(m)}}{m}\right)^m$$
since each side of the equation gives the present value of 1 to be paid at the end of the measurement period. Rearranging, we have
$$d=1-\left(1-\frac{d^{(m)}}{m}\right)^m$$
and solving this last equation for $d^{(m)}$ we find
$$d^{(m)}=m[1-(1-d)^{\frac{1}{m}}]=m(1-\nu^{\frac{1}{m}}).$$

Example 9.6
Find the present value of $1,000 to be paid at the end of six years at 6% per year payable in advance and convertible semiannually.

Solution.
The answer is
$$1,000\left(1-\frac{0.06}{2}\right)^{12}=\$693.84 \blacksquare$$
There is a close relationship between nominal rate of interest and nominal rate of discount. Since $1-d=\frac{1}{1+i}$, we conclude that
$$\left(1+\frac{i^{(m)}}{m}\right)^m=1+i=(1-d)^{-1}=\left(1-\frac{d^{(n)}}{n}\right)^{-n}. \tag{9.1}$$

If $m = n$ then the previous formula reduces to

$$\left(1+\frac{i^{(n)}}{n}\right)=\left(1-\frac{d^{(n)}}{n}\right)^{-1}.$$

Example 9.7
Find the nominal rate of discount convertible semiannually which is equivalent to a nominal rate of interest of 12% per year convertible monthly.

Solution.
We have

$$\left(1-\frac{d^{(2)}}{2}\right)^{-2}=\left(1+\frac{0.12}{12}\right)^{12}.$$

Solving for $d^{(2)}$ we find $d^{(2)} = 0.11591$ ■

Note that formula (9.1) can be used in general to find equivalent rates of interest or discount, either effective or nominal, converted with any desired frequency.

Example 9.8
Express $d^{(4)}$ as a function of $i^{(3)}$.

Solution.
We have

$$\left(1-\frac{d^{(4)}}{4}\right)^{-4}=\left(1+\frac{i^{(3)}}{3}\right)^{3}.$$

Thus,

$$1-\frac{d^{(4)}}{4}=\left(1+\frac{i^{(3)}}{3}\right)^{-\frac{3}{4}}$$

So

$$d^{(4)}=4\left[1-\left(1+\frac{i^{(3)}}{3}\right)^{-\frac{3}{4}}\right] \blacksquare$$

An analogous formula to $i - d = id$ holds for nominal rates of interest and discount as shown in the next example.

Example 9.9
Prove that

$$\frac{i^{(m)}}{m}-\frac{d^{(m)}}{m}=\frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m}$$

Solution.
We have

$$\left(1+\frac{i^{(m)}}{m}\right)=\left(1-\frac{d^{(m)}}{m}\right)^{-1}$$

which is equivalent to

$$\left(1+\frac{i^{(m)}}{m}\right)\cdot\left(1-\frac{d^{(m)}}{m}\right)=1.$$

Expanding we obtain

$$1-\frac{d^{(m)}}{m}+\frac{i^{(m)}}{m}-\frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m}=1.$$

Hence,

$$\frac{i^{(m)}}{m}-\frac{d^{(m)}}{m}=\frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m} \blacksquare$$

Example 9.10
Show that $i^{(m)}=d^{(m)}(1+i)^{\frac{1}{m}}$.

Solution.
We know that

$$\frac{i^{(m)}}{m}-\frac{d^{(m)}}{m}=\frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m}$$

Multiplying through by m and rearranging we find

$$i^{(m)}=d^{(m)}\left(1+\frac{i^{(m)}}{m}\right)=d^{(m)}(1+i)^{\frac{1}{m}} \blacksquare$$

Example 9.11
(a) On occasion, interest is convertible less frequently than once a year. Define $i^{(\frac{1}{m})}$ and $d^{(\frac{1}{m})}$ to be the nominal annual rates of interest and discount convertible once every m years. Find a formula analogous to formula (9.1) relating $i^{(\frac{1}{m})}$ and $d^{(\frac{1}{m})}$.
(b) Find the accumulated value of \$100 at the end of two years if the nominal annual rate of discount is 6%, convertible once every four years.

Solution.
(a) If $i^{(\frac{1}{m})}$ is convertible once every m years, then $m\cdot i^{(\frac{1}{m})}$ is the effective interest rate over m years. Thus, by definition of equivalent rates we can write

$$1+m\cdot i^{(\frac{1}{m})}=(1+i)^m.$$

Likewise, we have

$$1 - p \cdot d^{(\frac{1}{p})} = (1+i)^{-p}.$$

It follows that

$$(1 + m \cdot i^{(\frac{1}{m})})^{\frac{1}{m}} = (1 - p \cdot d^{(\frac{1}{p})})^{-\frac{1}{p}}.$$

Note that this formula can be obtained from formula (9.1) by replacing m and p by their reciprocals.
(b) If i is the annual effective interest rate than the accumulated value at the end of two years is

$$100(1+i)^2 = 100(1 - 4 \times 0.06)^{-\frac{2}{4}} = \$114.71 \blacksquare$$

Remark 9.1

Nominal rates of interest or discount are not relevant under simple interest and simple discount. For example, if we let $i^{(m)}$ be the nominal interest rate and i be the equivalent simple interest rate then at the end of one year we find $1 + i = 1 + i^{(m)}$ or $i = i^{(m)}$. Similarly, we have $d = d^{(m)}$.

Practice Problems

Problem 9.1
A man borrows \$1,000 at an interest rate of 24% per year compounded monthly. How much does he owe after 3 years?

Problem 9.2
If $i^{(6)} = 0.15$, find the equivalent nominal rate of interest convertible semiannually.

Problem 9.3
Suppose that \$100 is deposited into a savings account, earning at a discount rate of 0.15% biweekly, at the beginning of year 2006.
(a) Find the nominal annual discount rate.
(b) Find the effective annual discount rate.
(c) Find the amount of interest generated during the year.
(d) Find the equivalent effective annual interest rate.
(e) Find the equivalent nominal annual interest rate, convertible monthly.

Problem 9.4
Express $i^{(6)}$ as a function of $d^{(2)}$.

Problem 9.5
Given that $i^{(m)} = 0.1844144$ and $d^{(m)} = 0.1802608$. Find m.

Problem 9.6
It is known that

$$1 + \frac{i^{(n)}}{n} = \frac{1 + \frac{i^{(4)}}{4}}{1 + \frac{i^{(5)}}{5}}$$

Find n.

Problem 9.7
If $r = \frac{i^{(4)}}{d^{(4)}}$, express ν in terms of r.

Problem 9.8
You deposit \$1,000 into Account A and \$750 into Account B. Account A earns an effective annual interest rate of 5%. Account B earns interest at $i^{(4)}$. Ten years later, the two accounts have the same accumulated value. Find $i^{(4)}$. (Express as a percentage, to two decimal places.)

Problem 9.9 ‡
Eric deposits X into a savings account at time 0, which pays interest at a nominal rate of i, compounded semiannually. Mike deposits $2X$ into a different savings account at time 0, which pays simple interest at an annual rate of i. Eric and Mike earn the same amount of interest during the last 6 months of the 8th year. Calculate i.

Problem 9.10 ‡
Calculate the nominal rate of discount convertible monthly that is equivalent to a nominal rate of interest of 18.9% per year convertible monthly.

Problem 9.11
A bank credits interest on deposits quarterly at rate 2% per quarter. What is the nominal interest rate (per annum) for interest compounded quarterly? Find the annual effective rate of interest.

Problem 9.12
Interest on certain deposits is compounded monthly. Find the nominal rate of interest (per annum) for these deposits if the APR is 15%. Obtain the accumulation of an investment of $1,000 over a period of 6 months. How much interest has accrued by that time?

Problem 9.13
Suppose that the effective interest rate for $\frac{1}{5}$ year is 2%. Determine the equivalent nominal interest rate, compounded every 6 years.

Problem 9.14
Fund A earns interest at a nominal rate of 6% compounded monthly. Fund B earns interest at a nominal rate of discount compounded three times per year. The annual effective rates of interest earned by both funds are equivalent.
Calculate the nominal rate of discount earned by Fund B.

Problem 9.15
Treasury bills, or T−bills, are sold in terms ranging from a few days to 26 weeks. Bills are sold at a discount from their face value. For instance, you might pay $990 for a $1,000 bill. When the bill matures, you would be paid $1,000. The difference between the purchase price and face value is interest.
A bill for 100 is purchased for 96 three months before it is due. Find:
(a) The nominal rate of discount convertible quarterly earned by the purchaser.
(b) The annual effective rate of interest earned by the purchaser.

Problem 9.16
If $i^{(8)} = 0.16$, calculate $d^{(\frac{1}{2})}$.

Problem 9.17
A fund earns a nominal rate of interest of 6% compounded every two years. Calculate the amount that must be contributed now to have 1000 at the end of six years.

Problem 9.18
A deposit is made on January 1, 2004. The investment earns 6% compounded semi-annually. Calculate the monthly effective interest rate for the month of December 2004.

Problem 9.19
A deposit is made on January 1, 2004. The investment earns interest at a rate equivalent to an annual rate of discount of 6%.
Calculate the monthly effective interest rate for the month of December 2004.

Problem 9.20
A deposit is made on January 1, 2004. The investment earns interest at a rate equivalent to a rate of discount of 6% convertible quarterly.
Calculate the monthly effective interest rate for the month of December 2004.

Problem 9.21
Calculate the present value of $1,000 payable in 10 years using a discount rate of 5% convertible quarterly.

Problem 9.22
Calculate the accumulated value at the end of 3 years of 250 payable now assuming an interest rate equivalent to a discount rate of 12% convertible monthly.

Problem 9.23
Thomas pays 94 into a fund. Six months later, the fund pays Thomas 100.
Calculate the nominal rate of discount convertible semi-annually.

Problem 9.24
Thomas pays 94 into a fund. Six months later, the fund pays Thomas 100.
Calculate the annual effective rate of interest earned.

Problem 9.25
Find the present value of 5000, to be paid at the end of 25 months, at a rate of discount of 8% convertible quarterly:
(a) assuming compound discount throughout; (b) assuming simple discount during the final fractional period.

Problem 9.26 ‡
A bank offers the following certificates of deposit:

Term in years	Nominal annual interest rate (convertible quarterly)
1	4%
3	5%
5	5.65%

The bank does not permit early withdrawal. The certificates mature at the end of the term. During the next six years the bank will continue to offer these certificates of deposit with the same terms and interest rates.
An investor initially deposits \$10,000 in the bank and withdraws both principal and interest at the end of six years.
Calculate the maximum annual effective rate of interest the investor can earn over the 6-year period.

Problem 9.27
A bank offers the following certificates of deposit:

Term in years	Nominal annual interest rate (convertible semi-annually)
1	5%
2	6%
3	7%
4	8%

The bank does not permit early withdrawal. The certificates mature at the end of the term. During the next six years the bank will continue to offer these certificates of deposit. An investor plans to invest 1,000 in CDs. Calculate the maximum amount that can be withdrawn at the end of six years.

Problem 9.28
You are given
(1) Fund X accumulates at an interest rate of 8% compounded quarterly
(2) Fund Y accumulates at an interest rate of 6% compounded semiannually
(3) at the end of 10 years, the total amount in the two funds combined is 1000
(4) at the end of 5 years, the amount in Fund X is twice that in Fund Y
Calculate the total amount in the two funds at the end of 2 years.

Problem 9.29
A collection agency pays a doctor \$5,000 for invoices that the doctor has not been able to collect on. After two years, the collection agency has collected \$6,000 on the invoices. At what nominal rate of discount compounded monthly did the collection agency receive on this transaction?

Problem 9.30
A trust company offers guaranteed investment certificates paying $i^{(2)} = 8.9\%$ and $i^{(1)} = 9\%$. Which option yields the higher annual effective rate of interest?

10 Force of Interest: Continuous Compounding

Effective and nominal rates of interest and discount each measures interest over some interval of time. Effective rates of interest and discount measure interest over one full measurement period, while nominal rates of interest and discount measure interest over $m$ths of a period.
In this section we want to measure interest at any particular moment of time. This measure of interest is called the **force of interest.**
To start with, consider the case of a nominal compound interest $i^{(m)}$ converted m times a period. We can think of the force of interest, denoted by δ, as the limit of $i^{(m)}$ as the number of times we credit the compounded interest goes to infinity. That is,

$$\delta = \lim_{m\to\infty} i^{(m)}.$$

Letting i be the effective interest rate equivalent to $i^{(m)}$ we have

$$i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1] = \frac{(1+i)^{\frac{1}{m}} - 1}{\frac{1}{m}}.$$

Hence,

$$\delta = \lim_{m\to\infty} \frac{(1+i)^{\frac{1}{m}} - 1}{\frac{1}{m}}.$$

The above limit is of the form $\frac{0}{0}$ so that we can apply L'Hopital's rule to obtain

$$\delta = \lim_{m\to\infty} \frac{\frac{d}{dm}\left[(1+i)^{\frac{1}{m}} - 1\right]}{\frac{d}{dm}\left(\frac{1}{m}\right)} = \lim_{m\to\infty}[(1+i)^{\frac{1}{m}}\ln(1+i)] = \ln(1+i)$$

since $\lim_{m\to\infty}(1+i)^{\frac{1}{m}} = 1$.

Remark 10.1
The effective interest rate i can be written as a series expansion of δ :

$$i = e^{\delta} - 1 = \delta + \frac{\delta^2}{2!} + \cdots + \frac{\delta^n}{n!} + \cdots .$$

Similarly,

$$\delta = \ln(1+i) = i - \frac{i^2}{2} + \cdots + (-1)^n \frac{i^n}{n} + \cdots .$$

Example 10.1
Given the nominal interest rate of 12%, compounded monthly. Find the equivalent force of interest δ.

Solution.
The effective annual interest rate is

$$i = (1 + 0.01)^{12} - 1 \approx 0.1268250.$$

Hence, $\delta = \ln(1 + i) = \ln(1.1268250) \approx 0.119404$. ■

Remark 10.2
Intuitively, δ represents a nominal interest rate which is converted **continuously**, a notion of more theoretical than practical importance. Indeed, in theory the most important measure of interest is the force of interest. In practice, however, effective and nominal rates of interest tend to be used more frequently because most financial transactions involve discrete, not continuous, processes. However, δ can be used in practice as an approximation to interest converted very frequently, such as daily.

Having related δ and i immediately relates δ and the other measures of interested introduced in the previous sections. Indeed, we have the following important set of equalities

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = 1 + i = (1 - d)^{-1} = \left(1 - \frac{d^{(p)}}{p}\right)^{-p} = e^{\delta}.$$

Example 10.2
Using a constant force of interest of 4.2%, calculate the present value of a payment of \$1,000 to be made in 8 years' time.

Solution.
The present value is

$$1,000(1 + i)^{-8} = 1,000e^{-8\delta} = 1,000e^{-8\times 0.042} \approx \$714.62$$ ■

Example 10.3
A loan of \$3,000 is taken out on June 23, 1997. If the force of interest is 14%, find each of the following
(a) The value of the loan on June 23, 2002.
(b) The value of i.
(c) The value of $i^{(12)}$.

Solution.
(a) $3,000(1+i)^5 = 3,000e^{5\delta} = 3,000e^{0.7} \approx \$6,041.26$.
(b) $i = e^{\delta} - 1 = e^{0.14} - 1 \approx 0.15027$.
(c) We have

$$\left(1 + \frac{i^{(12)}}{12}\right)^{12} = 1 + i = e^{0.14}.$$

Solving for $i^{(12)}$ we find $i^{(12)} \approx 0.14082$ ■

Now, using the accumulation function of compound interest $a(t) = (1+i)^t$ we notice that

$$\delta = \ln(1+i) = \frac{\frac{d}{dt}a(t)}{a(t)}.$$

The definition of force of interest in terms of a compound interest accumulation function can be extended to any accumulation function. That is, for an accumulation function $a(t)$ we define the force of interest at time t by

$$\delta_t = \frac{a'(t)}{a(t)}.$$

Let's look closely at the expression on the right. From the definition of the derivative we have

$$\frac{\frac{d}{dt}a(t)}{a(t)} = \lim_{n\to\infty} \frac{\frac{a(t+\frac{1}{n})-a(t)}{a(t)}}{\frac{1}{n}}.$$

Now the expression

$$\frac{a(t+\frac{1}{n}) - a(t)}{a(t)}$$

is just the effective rate of interest over a very small time period $\frac{1}{n}$ so that

$$\frac{\frac{a(t+\frac{1}{n})-a(t)}{a(t)}}{\frac{1}{n}}$$

is the nominal annual interest rate converted n periods a year with each time period of length $\frac{1}{n}$ corresponding to that effective rate, which agrees with Remark 10.2.
Note that, in general, the force of interest can be a function of t. However, in the case of compound interest δ_t is a constant. Also, we notice that, since $A(t) = A(0)a(t)$, we can write

$$\delta_t = \frac{A'(t)}{A(t)}.$$

Example 10.4
Show that, for any amount function $A(t)$, we have

$$\int_0^n A(t)\delta_t dt = A(n) - A(0) = I_1 + I_2 + \cdots + I_n$$

Interpret this result verbally.

Solution.
We have

$$\int_0^n A(t)\delta_t dt = \int_0^n A'(t)dt = A(n) - A(0) = I_1 + I_2 + \cdots + I_n$$

where we used Example 2.6.
The term $A(n) - A(0)$ is the amount of interest earned over n measurement periods. The term $\delta_t dt$ represents the effective rate of interest over the infinitesimal "period of time" dt. Hence, $A(t)\delta_t dt$ is the amount of interest in this period and $\int_0^n A(t)\delta_t dt$ represents the total amount of interest earned over the entire n periods which is $A(n) - A(0)$ ■

Example 10.5
You are given that $A(t) = at^2 + bt + c$, for $0 \le t \le 2$, and that $A(0) = 100, A(1) = 110$, and $A(2) = 136$. Determine the force of interest at time $t = \frac{1}{2}$.

Solution.
The condition $A(0) = 100$ implies $c = 100$. From $A(1) = 110$ and $A(2) = 136$ we find the linear system of equations $a + b = 10$ and $4a + 2b = 36$. Solving this system we find $a = 8$ and $b = 2$. Hence, $A(t) = 8t^2 + 2t + 100$. It follows that

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{16t + 2}{8t^2 + 2t + 100}.$$

Hence,

$$\delta_{0.5} = \frac{10}{103} = 0.09709 ■$$

Example 10.6
Find δ_t in the case of simple interest, that is, when $a(t) = 1 + it$.

Solution.
From the definition of δ_t we have

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{i}{1 + it}$$

Note that δ_t is a decreasing function of t ■

We have now a method for finding δ_t given $a(t)$. What if we are given δ_t instead, and we wish to derive $a(t)$ from it?
From the definition of δ_t we can write

$$\frac{d}{dr}\ln a(r) = \delta_r.$$

Integrating both sides from 0 to t we obtain

$$\int_0^t \frac{d}{dr}\ln a(r)dr = \int_0^t \delta_r dr.$$

Hence,

$$\ln a(t) = \int_0^t \delta_r dr.$$

From this last equation we find

$$a(t) = e^{\int_0^t \delta_r dr}.$$

Also, we can derive an expression for $A(t)$. Since $A(t) = A(0)a(t)$, we can write

$$A(t) = A(0)e^{\int_0^t \delta_r dr}.$$

Example 10.7
A deposit of \$10 is invested at time 2 years. Using a force of interest of $\delta_t = 0.2 - 0.02t$, find the accumulated value of this payment at the end of 5 years.

Solution.
The accumulated value is

$$A(5) = 10\frac{a(5)}{a(2)} = 10e^{\int_2^5 (0.2-0.02t)dt} = 10e^{\left[0.2t-0.01t^2\right]_2^5} \approx \$14.77 ■$$

Note that a compound interest rate i implies a constant force of interest (equals to $\ln(1+i)$). The converse is also true as shown in the next example.

Example 10.8
Show that if $\delta_t = \delta$ for all t then $a(t) = (1+i)^t$ for some i.

Solution.
Since $\delta_t = \delta$ for all t, we obtain $\int_0^t \delta_r dr = t\delta$. Now, for all integer values $n \geq 1$ we have

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{e^{n\delta} - e^{(n-1)\delta}}{e^{(n-1)\delta}} = e^{\delta} - 1 = i.$$

In this case, $a(t) = e^{\delta t} = (e^{\delta})^t = (1+i)^t$ ■

The above example shows that a constant force of interest implies a constant effective interest rate. The converse of this result is not always true. See Problem 10.32.

Replacing the accumulation function by the discount function in the definition of δ_t we obtain the **force of discount**:

$$\delta'_t = -\frac{\frac{d}{dt}[a(t)]^{-1}}{[a(t)]^{-1}}.$$

The negative sign is necessary in order to make the force of discount a positive quantity since the discount function is a decreasing function of t.

It is possible to dispence with δ'_t and just use δ_t according to the following theorem.

Theorem 10.1
For all t we have $\delta'_t = \delta_t$.

Proof.
we have

$$\begin{aligned}\delta'_t = & -\frac{[a^{-1}(t)]'}{a^{-1}(t)}\\ = & \frac{a^{-2}(t)[a(t)]'}{a^{-1}(t)}\\ = & \frac{a^{-2}(t)a(t)\delta_t}{a^{-1}(t)}\\ = & \delta_t\end{aligned}$$

Example 10.9
Find the force of discount under a simple discount rate d.

Solution.
Recall that in the case of a simple discount the discount function is given by $[a(t)]^{-1} = 1 - dt$ for $0 \leq t < \frac{1}{d}$. Thus,

$$\begin{aligned}\delta_t =&\delta'_t = -\frac{\frac{d}{dt}[a(t)]^{-1}}{[a(t)]^{-1}}\\ =&-\frac{\frac{d}{dt}(1-dt)}{1-dt}\\ =&\frac{d}{1-dt}\end{aligned}$$

It follows that the force of interest is increasing with simple discount in contrast to simple interest where it is decreasing. See Example 10.6 ■

Example 10.10
Find δ'_t in the case of compound discount.

Solution.
We have

$$\delta'_t = -\frac{[(1-d)^t]'}{(1-d)^t} = -\ln(1-d) \blacksquare$$

Example 10.11
Show that $\lim_{m\to\infty} d^{(m)} = \delta$

Solution.
We have

$$\left(1 - \frac{d^{(m)}}{m}\right)^m = (1+i)^{-1} = e^{-\delta}.$$

Solving for $d^{(m)}$ we find $d^{(m)} = m[1 - e^{-\frac{\delta}{m}}]$. Using power series expansion of $e^{-\frac{\delta}{m}}$ we obtain

$$\begin{aligned}d^{(m)} =&m\left[1 - \left(1 + \left(-\frac{\delta}{m}\right) + \frac{1}{2!}\left(-\frac{\delta}{m}\right)^2 + \frac{1}{3!}\left(-\frac{\delta}{m}\right)^3 + \cdots\right)\right]\\ =&m\left[\frac{\delta}{m} - \frac{1}{2!}\frac{\delta^2}{m^2} + \frac{1}{3!}\frac{\delta^3}{m^3} - \cdots\right]\\ =&\delta - \frac{1}{2!}\frac{\delta^2}{m} + \frac{1}{3!}\frac{\delta^3}{m^2} - \cdots \to \delta \; as \; m \to \infty \blacksquare\end{aligned}$$

Example 10.12
An investor invests \$1,000 into an investment account at time 1. The interest is computed based on the following force of interest:

$$\delta_t = \begin{cases} 0.02t & 0 \leq t < 3 \\ 0.025 & t \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the amount of interest generated between time 2.5 and 3.5.
(b) Calculate effective discount rate during the third period, i.e. d_3.

Solution.
(a) We know that $A(t) = 1000\frac{a(t)}{a(1)} = 1000e^{\int_1^t \delta_r dr}$. Thus, the amount of interest generated between time 2.5 and 3.5 is

$$A(3.5) - A(2.5) = 1000\left(e^{\int_1^{3.5}\delta_t dt} - e^{\int_1^{2.5}\delta_t dt}\right).$$

But

$$\int_1^{3.5}\delta_t dt = \int_1^3 0.02tdt + \int_3^{3.5} 0.025dt = 0.0925$$

and

$$\int_1^{2.5}\delta_t dt = \int_1^{2.5} 0.02tdt = 0.0525$$

Hence,

$$A(3.5) - A(2.5) = 1000(e^{0.0925} - e^{0.0525}) \approx \$43.01.$$

(b) $d_3 = \frac{A(3)-A(2)}{A(3)} = \frac{e^{0.08}-e^{0.03}}{e^{0.08}} \approx 4.877\%$ ■

Example 10.13
Find an expression of t in terms of δ so that $f(t) = (1+it) - (1+i)^t$ is maximum.

Solution.
Since $f'(t) = i - (1+i)^t \ln(1+i) = i - \delta(1+i)^t$ we see that $f'(t) = 0$ when $(1+i)^t = \frac{i}{\delta}$. Solving for t we find $t = \frac{\ln i - \ln\delta}{\ln(1+i)} = \frac{\ln i - \ln\delta}{\delta}$. Now, $f''(t) = -\delta^2(1+i)^t < 0$ so that the critical point of f is a maximum ■

Example 10.14
On March 15, 2003, a student deposits X into a bank account. The account is credited with simple interest where $i = 7.5\%$

On the same date, the student's professor deposits X into a different bank account where interest is credited at a force of interest

$$\delta_t = \frac{2t}{t^2+k}, \quad t \geq 0$$

From the end of the fourth year until the end of the eighth year, both accounts earn the same dollar amount of interest. Calculate k.

Solution.
The interest earned by the student is

$$X[1+0.075(8)] - X[1+0.075(4)] = 0.3X$$

The interest earned by the professor is

$$\begin{aligned} Xe^{\int_0^8 \delta_t dt} - Xe^{\int_0^4 \delta_t dt} &= Xe^{\int_0^8 \frac{2t}{t^2+k} dt} - Xe^{\int_0^4 \frac{2t}{t^2+k} dt} \\ &= Xe^{\left[\ln(t^2+k)\right]_0^8} - Xe^{\left[\ln(t^2+k)\right]_0^4} \\ &= X\frac{64+k}{k} - X\frac{16+k}{k} = \frac{48}{k}X. \end{aligned}$$

Thus, $0.3X = \frac{48}{k}X \Rightarrow k = \frac{48}{0.3} = 160$ ■

Example 10.15
Show that $d < d^{(m)} < \delta < i^{(m)} < i, \quad m > 1.$

Solution.
We have

$$\frac{d}{dm}(i^{(m)}) = e^{\frac{\delta}{m}}\left(1 - \frac{\delta}{m} - e^{-\frac{\delta}{m}}\right).$$

From the inequality $1 - x \leq e^{-x}$ for $x \geq 0$ we conclude that

$$\frac{d}{dm}(i^{(m)}) = e^{\frac{\delta}{m}}\left(1 - \frac{\delta}{m} - e^{-\frac{\delta}{m}}\right) < 0.$$

That is, $i^{(m)}$ is a decreasing function of m. Since $i^{(1)} = i$ and $i^{(\infty)} = \delta$ we obtain

$$\delta < i^{(m)} < i.$$

Similarly,

$$\frac{d}{dm}(d^{(m)}) = e^{-\frac{\delta}{m}}(e^{\frac{\delta}{m}} - 1 - \frac{\delta}{m}) > 0$$

so that $d^{(m)}$ is an increasing function of m. Since $d^{(1)} = d$ and $d^{(\infty)} = \delta$ we can write

$$d < d^{(m)} < \delta.$$

Combining these inequalities we find

$$d < d^{(m)} < \delta < i^{(m)} < i, \quad m > 1 \blacksquare$$

Practice Problems

Problem 10.1
A deposit of \$500 is invested at time 5 years. The constant force of interest is 6% per year. Determine the accumulated value of the investment at the end of 10 years.

Problem 10.2
If the constant force of interest is 6%, what is the corresponding annual effective rate of interest?

Problem 10.3
Assume that the force of interest varies with time and is given by $\delta_t = a + \frac{b}{t}$. Find the formula for the accumulation of one unit money from time t_1 to time t_2.

Problem 10.4
You need \$500 on Jan 1, 2012. To save for this amount, you invest x on Jan 1, 2008 and $2x$ on July 1, 2008. The force of interest is $\delta_t = 0.02t$ where t is 0 on Jan 1, 2008. Find x.

Problem 10.5
At a constant force of interest, \$200 accumulates to \$240 over the course of 8 years. Find the force of interest δ.

Problem 10.6 ‡
Bruce deposits 100 into a bank account. His account is credited interest at a nominal rate of interest of 4% convertible semiannually. At the same time, Peter deposits 100 into a separate account. Peter's account is credited interest at a force of interest of δ. After 7.25 years, the value of each account is the same. Calculate δ.

Problem 10.7 ‡
Bruce deposits 100 into a bank account. His account is credited interest at a nominal rate of interest i convertible semiannually. At the same time, Peter deposits 100 into a separate account. Peter's account is credited interest at a force of interest of δ. After 7.25 years, the value of each account is 200. Calculate $i - \delta$.

Problem 10.8 ‡
At time $t = 0$, 1 is deposited into each of Fund X and Fund Y. Fund X accumulates at a force of interest $\delta_t = \frac{t^2}{k}$. Fund Y accumulates at a nominal rate of discount of 8% per annum convertible semiannually.
At time $t = 5$, the accumulated value of Fund X equals the accumulated value of Fund Y.
Determine k.

Problem 10.9 ‡
At time 0, K is deposited into Fund X, which accumulates at a force of interest $\delta_t = 0.006t^2$. At time m, $2K$ is deposited into Fund Y, which accumulates at an annual effective interest rate of 10%. At time n, where $n > m$, the accumulated value of each fund is $4K$. Determine m.

Problem 10.10
Suppose \$2,000 is invested for 3 years with a constant force of interest equal to 9%.
(a) Find the accumulation function.
(b) Find the effective rate of interest.
(c) Find the value of the investment after 3 years.

Problem 10.11 ‡
Ernie makes deposits of 100 at time 0, and X at time 3 . The fund grows at a force of interest $\delta_t = \frac{t^2}{100}$, $t > 0$. The amount of interest earned from time 3 to time 6 is X. Calculate X.

Problem 10.12
If the force of interest δ is 0.1, obtain the nominal rate of interest (per annum) on deposits compounded (a) monthly; (b) quarterly; (c) annually.

Problem 10.13
The force of interest depends upon time, with $\delta(t) = \frac{1}{10(1+t)}$, $0 \le t \le 4$. Find the accumulated amount after four years for an initial investment of \$1,000 at time zero.

Problem 10.14
The force of interest δ is constant. The corresponding nominal rate of discount when interest is compounded p-thly is $d^{(p)}$. Express $d^{(p)}$ in terms of p. Show that $d^{(p)}$ is an increasing function of p.

Problem 10.15 ‡
Tawny makes a deposit into a bank account which credits interest at a nominal interest rate of 10% per annum, convertible semiannually. At the same time, Fabio deposits 1,000 into a different bank account, which is credited with simple interest. At the end of 5 years, the forces of interest on the two accounts are equal, and Fabio's account has accumulated to Z. Determine Z.

Problem 10.16
Fund A earns interest at a nominal rate of interest of 12% compounded quarterly. Fund B earns interest at a force of interest δ. John invested \$1,000 in each fund five years ago. Today, the amount in Fund A is equal to the amount in Fund B. Calculate δ.

Problem 10.17
A fund earns interest at a force of interest of $\delta = 0.01t$. Calculate the value at the end of 10 years of 8,000 invested at the end of 5 years.

Problem 10.18
A fund earns interest at a force of interest of $\delta = 0.01t$. Calculate the effective rate of interest in the 10th year.

Problem 10.19
Calculate the effective annual interest rate equivalent to a nominal rate of discount of 6% compounded continuously.

Problem 10.20
Fund A accumulates at a simple interest rate of 10%. Find B accumulates at a simple discount rate of 5%. Find the point in time at which the force of interest on the two funds are equal.

Problem 10.21
An investment is made for one year in a fund with $a(t) = a + bt + ct^2$. The nominal rate of interest earned during the first half of the year is 5% convertible semi-annually. The effective rate of interest earned for the entire year is 7%. Find $\delta_{0.5}$.

Problem 10.22
If $a(t) = 1 + .01(t^2 + t)$, calculate δ_5.

Problem 10.23
You are given that $\delta = 0.05$. Calculate the amount that must be invested at the end of 10 years to have an accumulated value at the end of 30 years of \$1,000.

Problem 10.24
Calculate k if a deposit of 1 will accumulate to 2.7183 in 10 years at a force of interest given by:

$$\delta_t = \begin{cases} kt & 0 < t \leq 5 \\ 0.04kt^2 & t > 5 \end{cases}$$

Problem 10.25
A deposit is made on January 1, 2004. The investment earns interest at a constant force of interest of 6%.
Calculate the monthly effective interest rate for the month of December 2004.

Problem 10.26
A fund earns interest at a constant force of interest of δ. Kathy invested 10,000 in the fund. Twenty two years later, Kathy has 30,042. Calculate δ.

Problem 10.27
A fund earns interest at a force of interest of $\delta_t = kt$. Ryan invests 1,000 at time $t = 0$. After 8 years, Ryan has 1,896.50 . Calculate k.

Problem 10.28
You are given that $i^{(12)} = 0.09$. Calculate δ.

Problem 10.29
You are given that $d^{(2)} = 0.04$. Calculate δ.

Problem 10.30 ‡
At time 0, deposits of 10,000 are made into each of Fund X and Fund Y. Fund X accumulates at an annual effective interest rate of 5%. Fund Y accumulates at a simple interest rate of 8%. At time t, the forces of interest on the two funds are equal. At time t, the accumulated value of Fund Y is greater than the accumulated value of Fund X by Z. Determine Z.

Problem 10.31
Find an expression for the fraction of a period at which the excess of present values computed at simple discount over compound discount is a maximum. Hint: See Example 10.13.

Problem 10.32
Consider the following problem: Invest 1 at a compound interest i for integral periods and at a simple interest for fractional periods.
(a) Show that $i_k = i$ for $k = 1, 2, \cdots, n$.
(b) Show that $\delta_{t_1} \neq \delta_{t_2}$ for $k < t_1 < t_2 < k+1,\ k = 1, 2, \cdots n-1$.
Thus, a constant effective rate of interest i does not necessarily imply a constant force of interest.

Problem 10.33
Which of the following are true?
(I) $\frac{d}{dd}(i) = \nu^{-2}$

(II) $\frac{d}{di}(i^{(m)}) = \nu^{-(\frac{m-1}{m})}$

(III) $\frac{d}{d\delta}(i) = 1 + i$.

Problem 10.34
You are given that $\delta_t = \frac{0.2t}{1+0.1t^2}$. Calculate i_2.

Problem 10.35
Simplify the following expression $\left(\frac{d}{d\nu}\delta\right)\left(\frac{d}{di}d\right)$.

Problem 10.36
An investment is made in an account in which the amount function is given by $A(t) = t^2 + t + 2.25$, $0 \le t \le 4$. Compute the ratio $\frac{i_3}{\delta_2}$.

Problem 10.37
Fund A accumulates at a force of interest

$$\frac{0.05}{1 + 0.05t}$$

at time $t \ge 0$. Fund B accumulates at a force of interest 0.05. You are given that the amount in Fund A at time zero is 1,000, the amount in Fund B at time zero is 500, and that the amount in Fund C at any time t is equal to the sum of the amount in Fund A and Fund B. Fund C accumulates at force of interest δ_t. Find δ_2.

Problem 10.38
Show that

$$\frac{d}{dt}\delta_t = \frac{A''(t)}{A(t)} - \delta_t^2.$$

Problem 10.39
Find δ_t if $A(t) = K2^t3^{t^2}5^{2^t}$.

Problem 10.40
Show that $\int_0^n \delta_t dt = \ln a(n)$.

11 Time Varying Interest Rates

In this section we consider situations involving varying interest. The first involves a continuously varying force of interest δ_t. In this case, the accumulated value at time t is given by

$$A(t) = A(0)e^{\int_0^t \delta_r dr}.$$

Example 11.1
Find the accumulated value of \$200 invested for 5 years if the force of interest is $\delta_t = \frac{1}{8+t}$.

Solution.
The accumulated value is $A(5) = 200a(5) = 200e^{\int_0^5 \frac{dt}{8+t}} = 200e^{[\ln(8+t)]_0^5} = 200 \times \frac{13}{8} = \$325$ ■

The second situation involves changes in the effective rate of interest over a period of time. Letting i_n denote the effective rate of interest during the n^{th} period from the date of investment, the accumulated value for integral t is given by

$$a(t) = (1+i_1)(1+i_2)\cdots(1+i_t) \tag{11.1}$$

and the present value is given by

$$(a(t))^{-1} = (1+i_1)^{-1}(1+i_2)^{-1}\cdots(1+i_n)^{-1}. \tag{11.2}$$

Example 11.2
Find the accumulated value of \$500 invested for 9 years if the rate of interest is 5% for the first 3 years, 5.5% for the second 3 years, and 6.25% for the third 3 years.

Solution.
The accumulated value is

$$500(1+0.05)^3(1+0.055)^3(1+0.0625)^3 = \$815.23 \blacksquare$$

Formulas (11.1) and (11.2) can be extended to include nominal rates of interest or discount. We illustrate this in the next example.

Example 11.3
A fund will earn a nominal rate of interest of 5% compounded quarterly during the first two years, a nominal rate of discount of 4% compounded monthly during years 3 and 4, and a constant force of interest of 3% during the fifth and sixth year.
Calculate the amount that must be invested today in order to accumulate 5,000 after 6 years.

Solution.
The amount that must be invested is $5000\left(1+\frac{0.05}{4}\right)^{-8}\left(1-\frac{0.04}{12}\right)^{24}e^{-0.03\times 2} = \$3,935.00$ ■

A common question involving varying interest is the question of finding an equivalent level rate to the rate that vary. Moreover, the answer depends on the period of time chosen for the comparison . We illustrate this point in the next example.

Example 11.4
(a) Find the effective interest rate over the first four-year period from now if the effective rate of interest is 5% for the first three years from now, 4.5% for the next three years, and 4% for the last three years.
(b) What about over the six-year period from now?

Solution.
(a) We have $(1+i)^4 = (1.05)^3(1.045)$. Solving this equation for i we find

$$i = [(1.05)^3(1.045)]^{\frac{1}{4}} - 1 = 4.87\%$$

(b) We have $(1+i)^6 = (1.05)^3(1.045)^3$. Solving this equation for i we find

$$i = [(1.05)^3(1.045)^3]^{\frac{1}{6}} - 1 = 4.75\% \blacksquare$$

Example 11.5
Find the accumulated value of 1 at the end of n periods where the effective rate of interest for the kth period, $1 \le k \le n$, is defined by

$$i_k = (1+r)^k(1+i) - 1.$$

Solution.
We know that $a(n) = (1+i_1)(1+i_2)\cdots(1+i_n) = (1+r)(1+i)(1+r)^2(1+i)\cdots(1+r)^n(1+i) = (1+r)^{1+2+\cdots+n}(1+i)^n = (1+r)^{\frac{n(n+1)}{2}}(1+i)^n$ ■

Practice Problems

Problem 11.1
Find the effective interest rate over a three-year period which is equivalent to an effective rate of discount of 8% the first year, 7% the second year, and 6% the third year.

Problem 11.2
Find the accumulated value of 1 at the end of n years if the force of interest is $\delta_t = \frac{1}{1+t}$.

Problem 11.3
Find the accumulated value of 1 at the end of 19 years if the force of interest is $\delta_t = 0.04(1+t)^{-2}$.

Problem 11.4
If $\delta_t = 0.01t,\ 0 \le t \le 2$, find the equivalent annual effective rate of interest over the interval $0 \le t \le 2$.

Problem 11.5
An investment account earns 4% the first year, 5% the second year, and X% the third year. Find X if the three year average interest rate is 10%.

Problem 11.6
An investment of \$500 was made three years ago. For the first six months the annual effective rate of interest paid on the account was 5%, but it then increased to 7% and did not change over the following two and a half years. Find the present accumulation of the investment.

Problem 11.7
The annual effective interest rate for a given year is determined by the function: $i_n = 0.02n$, where $n = 1, 2, 3, \cdots$. Brad's initial investment of \$400 earns interest every year according to the preceding. Find the accumulated value of his investment after six years.

Problem 11.8 ‡
At a force of interest $\delta_t = \frac{2}{(k+2t)}$
(i) a deposit of 75 at time $t = 0$ will accumulate to X at time $t = 3$; and
(ii) the present value at time $t = 3$ of a deposit of 150 at time $t = 5$ is also equal to X.
Calculate X.

Problem 11.9 ‡
Jennifer deposits 1000 into a bank account. The bank credits interest at a nominal annual rate of i compounded semi-annually for the first 7 years and a nominal annual rate of $2i$ convertible quarterly for all years thereafter. The accumulated amount in the account after 5 years is X. The accumulated amount in the account at the end of 10.5 years is 1,980. Calculate X.

Problem 11.10
Suppose $\delta_t = \frac{t^3}{100}$. Find $\frac{1}{a(3)}$.

Problem 11.11
In Fund X money accumulates at a force of interest

$$\delta_t = 0.01t + 0.1, \quad 0 \le t \le 20.$$

In Fund Y money accumulates at an annual effective interest rate i. An amount of 1 is invested in each fund for 20 years. The value of Fund X at the end of 20 years is equal to the value of Fund Y at the end of 20 years. Calculate the value of Fund Y at the end of 1.5 years.

Problem 11.12
If the effective rate of discount in year k is equal to $0.01k + 0.06$ for $k = 1, 2, 3$, find the equivalent rate of simple interest over the three-year period.

Problem 11.13
A savings and loan association pays 7% effective on deposits at the end of each year. At the end of every 3 years a 2% bonus is paid on the balance at that time. Find the effective rate of interest earned by an investor if the money is left on deposit
(a) Two years.
(b) Three years.
(c) Four years.

Problem 11.14
On July 1, 1999 a person invested 1,000 in a fund for which the force of interest at time t is given by $\delta_t = .02(3 + 2t)$ where t is the number of years since January 1, 1999.
Determine the accumulated value of the investment on January 1, 2000.

Problem 11.15
You are given that $\delta_t = \frac{t}{100}$. Calculate the present value at the end of the 10 year of an accumulated value at the end of 15 years of \$1,000.

Problem 11.16
Calculate k if a deposit of 1 will accumulate to 2.7183 in 10 years at a force of interest given by:

$$\delta_t = \begin{cases} kt & 0 < t \le 5 \\ 0.04kt^2 & t > 5 \end{cases}$$

Problem 11.17
The annual effective interest rate for year t is $i_t = \frac{1}{(10+t)}$.
Calculate the current value at the end of year 2 of a payment of 6,000 at the end of year 7.

Problem 11.18
A fund pays a nominal rate of 12%. The nominal rate is compounded once in year 1, twice in year 2, 3 times in year 3, etc.
Calculate the amount that must be invested today in order to accumulate 5,000 after 6 years.

Problem 11.19
You are given $\delta_t = \frac{2}{t-1}$, for $2 \leq t \leq 20$. For any one year interval between n and $n+1$, with $2 \leq n \leq 9$, calculate $d_{n+1}^{(2)}$.

Problem 11.20
Investment X for 100,000 is invested at a nominal rate of interest j convertible semi-annually. After four years it accumulates to 214,358.88. Investment Y for 100,000 is invested at a nominal rate of discount k convertible quarterly. After two years, it accumulates to 232,305.73. Investment Z for 100,000 is invested at an annual effective rate of interest equal to j in year one and an annual effective rate of discount equal to k in year two. Calculate the value of investment Z at the end of two years.

Problem 11.21
A bank agrees to lend John 10,000 now and X three years later in exchange for a single repayment of 75,000 at the end of 10 years. The bank charges interest at an annual effective rate of 6% for the first 5 years and at a force of interest $\delta_t = \frac{1}{t+1}$ for $t \geq 5$. Determine X.

Problem 11.22
John invests 1000 in a fund which earns interest during the first year at a nominal rate of K convertible quarterly. During the 2nd year the fund earns interest at a nominal discount rate of K convertible quarterly. At the end of the 2nd year, the fund has accumulated to 1173.54. Calculuate K.

Problem 11.23
At time 0, 100 is deposited into Fund$X$ and also into Fund Y. Fund$X$ accumulates at a force of interest $\delta_t = 0.5(1+t)^{-2}$. Fund Y accumulates at an annual effective interest rate of i. At the end of 9 years, the accumulated value of Fund X equals the accumulated value of Fund Y. Determine i.

Problem 11.24 ‡
An investor deposits 1000 on January 1 of year x and deposits 1000 on January 1 of year $x+2$ into a fund that matures on January 1 of year $x+4$. The interest rate on the fund differs every year and is equal to the annual effective rate of growth of the gross domestic product (GDP) during the fourth quarter of the previous year.

The following are the relevant GDP values for the past 4 years.

Year	$x-1$	x	$x+1$	$x+2$
Quarter III	800.0	850.0	900.0	930.0
Quarter IV	808.0	858.5	918.0	948.6

What is the internal rate of return earned by the investor over the 4-year period?

Problem 11.25
A deposit of 10,000 is made into a fund at time $t=0$. The fund pays interest at a nominal rate of discount of d compounded quarterly for the first two years. Beginning at time $t=2$, interest is credited using a nominal rate of interest of 8% compounded quarterly. At time $t=5$, the accumulated value of the fund is 14,910. Calculate d.

Problem 11.26
Amin deposits 10,000 in a bank. During the first year the bank credits an annual effective rate of interest i. During the second year the bank credits an annual effective rate of interest $(i-5\%)$. At the end of two years she has 12,093.75 in the bank. What would Amin have in the bank at the end of three years if the annual effective rate of interest were $(i+9\%)$ for each of the three years?

Problem 11.27
Fund X starts with 1,000 and accumulates with a force of interest

$$\delta_t = \frac{1}{15-t}$$

for $0 \le t < 15$. Fund Y starts with 1,000 and accumulates with an interest rate of 8% per annum compounded semi-annually for the first three years and an effective interest rate of i per annum thereafter. Fund X equals Fund Y at the end of four years. Calculate i.

Problem 11.28
Amin puts 100 into a fund that pays an effective annual rate of discount of 20% for the first two years and a force of interest of rate

$$\delta_t = \frac{2t}{t^2+8} < \quad 2 \le t \le 4$$

for the next two years. At the end of four years, the amount in Amin's account is the same as what it would have been if he had put 100 into an account paying interest at the nominal rate of i per annum compounded quarterly for four years. Calculate i.

Problem 11.29
On January 1, 1980, Jack deposited 1,000 into Bank X to earn interest at the rate of j per annum compounded semi-annually. On January 1, 1985, he transferred his account to Bank Y to earn interest at the rate of k per annum compounded quarterly. On January 1, 1988, the balance at Bank Y is 1,990.76. If Jack could have earned interest at the rate of k per annum compounded quarterly from January 1, 1980 through January 1, 1988, his balance would have been 2,203.76. Calculate the ratio $\frac{k}{j}$.

12 Equations of Value and Time Diagrams

Interest problems generally involve four quantities: principal(s), investment period length(s), interest rate(s), accumulated value(s). If any three of these quantities are known, then the fourth quantity can be determined. In this section we introduce equations that involve all four quantities with three quantities are given and the fourth to be found.
In calculations involving interest, the value of an amount of money at any given point in time depends upon the time elapsed since the money was paid in the past or upon time which will elapse in the future before it is paid. This principle is often characterized as the recognition of the **time value of money.** We assume that this principle reflects only the effect of interest and does not include the effect of inflation. Inflation reduces the the purchasing power of money over time so investors expect a higher rate of return to compensate for inflation. As pointed out, we will neglect the effct of inflation when applying the above mentioned principle.
As a consequence of the above principle, various amounts of money payable at different points in time cannot be compared until all the amount are accumulated or discounted to a common date, called the **comparison date,** is established. The equation which accumulates or discounts each payment to the comparison date is called the **equation of value.**
One device which is often helpful in the solution of equations of value is the **time diagram**. A time diagram is a one-dimensional diagram where the only variable is time, shown on a single coordinate axis. We may show above or below the coordinate of a point on the time-axis, values of money intended to be associated with different funds. A time diagram is not a formal part of a solution, but may be very helpful in visualizing the solution. Usually, they are very helpful in the solution of complex probems.

Example 12.1
In return for a payment of $1,200 at the end of 10 years, a lender agrees to pay $200 immediately, $400 at the end of 6 years, and a final amount at the end of 15 years. Find the amount of the final payment at the end of 15 years if the nominal rate of interest is 9% converted semiannually.

Solution.
The comparison date is chosen to be $t = 0$. The time diagram is given in Figure 12.1.

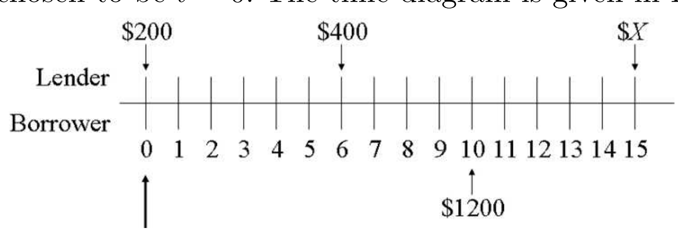


Figure 12.1

The equation of value is

$$200 + 400(1 + 0.045)^{-12} + X(1 + 0.045)^{-30} = 1200(1 + 0.045)^{-20}.$$

Solving this equation for X we find $X \approx \$231.11$ ■

With compound interest, an equation of value will produce the same answer for an unknown value regardless of what comparison date is selected. This is due to the fact that multiplying an equation by a power of an expression yields an equivalent equation. We illustrate this in the next example.

Example 12.2
In return for a promise to receive \$600 at the end of 8 years, a person agrees to pay \$100 at once, \$200 at the end of 5 years, and to make further payment at the end of 10 years. Find the payment at the end of 10 years if the nominal interest rate is 8% convertible semi-annually.

Solution.
The equation of value at $t = 0$ is

$$100 + 200(1 + 0.04)^{-10} + X(1 + 0.04)^{-20} = 600(1 + 0.04)^{-16}.$$

Solving for X we find $X \approx 186.75$.
The equation of value at $t = 10$ is

$$100(1 + 0.04)^{20} + 200(1 + 0.04)^{10} + X = 600(1 + 0.04)^{4}.$$

Solving for X we find $X \approx 186.75$. Note that by multiplying the last equation by $(1 + 0.04)^{-20}$ one gets the equation of value at $t = 0$ ■

Whereas the choice of a comparison date has no effect on the answer obtained with compound interest, the same cannot be said of simple interest or simple discount.

Example 12.3
Find the amount to be paid at the end of 10 years which is equivalent to two payments of 100 each, the first to be paid immediately, and the second to be paid at the end of 5 years. Assume 5% simple interest is earned from the date each payment is made, and use a comparison date
1. The end of 10 years.
2. The end of 15 years.

Solution.
1. With a comparison date at $t = 10$, the payment at the end of 10 years will be

$$100(1 + 10 \times 0.05) + 100(1 + 5 \times 0.05) = 275.$$

2. With a comparison date at $t = 15$, the amount P that must be paid at the end of 10 years satisfies the equation of value

$$100(1 + 15 \times 0.05) + 100(1 + 10 \times 0.05) = P(1 + 5 \times 0.05)$$

which implies that $P = 260$ ■

Example 12.4
Investor A deposits 1,000 into an account paying 4% compounded quarterly. At the end of three years, he deposits an additional 1,000. Investor B deposits X into an account with force of interest $\delta_t = \frac{1}{6+t}$. After five years, investors A and B have the same amount of money. Find X.

Solution.
Consider investor A's account first. The initial 1,000 accumulates at 4% compounded quarterly for five years; the accumulated amount of this piece is

$$1,000\left(1 + \frac{0.04}{4}\right)^{4\times 5} = 1000(1.01)^{20}.$$

The second 1,000 accumulates at 4% compounded quarterly for two years, accumulating to

$$1,000\left(1 + \frac{0.04}{4}\right)^{4\times 2} = 1000(1.01)^{8}.$$

The value in investor A's account after five years is

$$A = 1000(1.01)^{20} + 1000(1.01)^{8}.$$

The accumulated amount of investor B's account after five years is given by

$$B = Xe^{\int_0^5 \frac{dt}{6+t}} = Xe^{\ln\left(\frac{11}{6}\right)} = \frac{11}{6}X.$$

The equation of value at time $t = 5$ is

$$\frac{11}{6}X = 1000(1.01)^{20} + 1000(1.01)^{8}.$$

Solving for X we find $X \approx \$1,256.21$ ■

Practice Problems

Problem 12.1
In return for payments of \$5,000 at the end of 3 years and \$4,000 at the end of 9 years, an investor agrees to pay \$1500 immediately and to make an additional payment at the end of 2 years. Find the amount of the additional payment if $i^{(4)} = 0.08$.

Problem 12.2
At a certain interest rate the present values of the following two payment patterns are equal:
(i) 200 at the end of 5 years plus 500 at the end of 10 years;
(ii) 400.94 at the end of 5 years.
At the same interest rate 100 invested now plus 120 invested at the end of 5 years will accumulate to P at the end of 10 years. Calculate P.

Problem 12.3
An investor makes three deposits into a fund, at the end of 1, 3, and 5 years. The amount of the deposit at time t is $100(1.025)^t$. Find the size of the fund at the end of 7 years, if the nominal rate of discount convertible quarterly is $\frac{4}{41}$.

Problem 12.4 ‡
Brian and Jennifer each take out a loan of X. Jennifer will repay her loan by making one payment of 800 at the end of year 10. Brian will repay his loan by making one payment of 1,120 at the end of year 10. The nominal semi-annual rate being charged to Jennifer is exactly one-half the nominal semi-annual rate being charged to Brian. Calculate X.

Problem 12.5
Fund A accumulates at 6% effective, and Fund B accumulates at 8% effective. At the end of 20 years the total of the two funds is 2,000. At the end of 10 years the amount in Fund A is half that in Fund B. What is the total of the two funds at the end of 5 years?

Problem 12.6
Louis has an obligation to pay a sum of \$3,000 in four years from now and a sum of \$5,000 in six years from now. His creditor permits him to discharge these debts by paying \$X in two years from now, \$1000 in three years from now, and a final payment of \$2X in nine years from now. Assuming an annual effective rate of interest of 6%, find X.

Problem 12.7
Every Friday in February (7, 14, 21,28) Vick places a 1,000 bet, on credit, with his off-track bookmaking service, which charges an effective weekly interest rate of 8% on all credit extended.

Vick looses each bet and agrees to repay his debt in four installments to be made on March 7, 14, 21, and 28. Vick pays 1,100 on March 7, 14, and 21. How much must Vick pay on March 28 to completely repay his debt?

Problem 12.8
A borrower is repaying a loan by making payments of 1,000 at the end of each of the next 3 years. The interest rate on the loan is 5% compounded annually. What payment could the borrower make at the end of the first year in order to extinguish the loan?

Problem 12.9
An investor purchases an investment which will pay 2,000 at the end of one year and 5,000 at the end of four years. The investor pays 1,000 now and agrees to pay X at the end of the third year. If the investor uses an interest rate of 7% compounded annually, what is X?

Problem 12.10
Today is New Year's Day. In return for payments of 1,500 at the end of January, February, and March, and of 3,000 at the end of May, July, and September, an investor agrees to pay now the total value of the 6 payments, and to either make or receive an additional payment at the end of December. Find the amount of that additional payment if it is known that the nominal annual interest rate is 6%, compounded monthly.

Problem 12.11
A debt of 7,000 is due at the end of 5 years. If 2,000 is paid at the end of 1 year, what single payment should be made at the end of the 2nd year to liquidate the debt, assuming interest at the rate of 6.5% per year, compounded quarterly.

Problem 12.12
George agrees to buy his brother's car for 7,000. He makes a down payment of 4,000 now, and agrees to pay two equal payments, one at the end of 6 months, and the other at the end of a year. If interest is being charged at 5% per annum effective, how large should each of the equal payments be?

Problem 12.13 ‡
Jeff deposits 10 into a fund today and 20 fifteen years later. Interest is credited at a nominal discount rate of d compounded quarterly for the first 10 years, and at a nominal interest rate of 6% compounded semiannually thereafter. The accumulated balance in the fund at the end of 30 years is 100. Calculate d.

Problem 12.14 ‡
The parents of three children, ages 1, 3, and 6, wish to set up a trust fund that will pay X to each child upon attainment of age 18, and Y to each child upon attainment of age 21. They will establish the trust fund with a single investment of Z. Find the equation of value for Z.

Problem 12.15
An investment fund is established at time 0 with a deposit of 5000. 1000 is added at the end of 4 months, and an additional 4000 is added at the end of 8 months. No withdrawals are made. The fund value, including interest, is 10560 at the end of 1 year. The force of interest at time t is $\frac{k}{1+(1-t)k}$, for $0 \leq t \leq 1$. Determine k.

Problem 12.16
A loan of 1000 is made at an interest rate of 12% compounded quarterly. The loan is to be repaid with three payments: 400 at the end of the first year, 800 at the end of the fifth year, and the balance at the end of the tenth year. Calculate the amount of the final payment.

Problem 12.17
You are given $\delta_t = \frac{0.5}{1+t}$ for $t > 0$. A payment of 400 at the end of 3 years and 800 at the end of 15 years has the same present value as a payment of 300 at the end of 8 years and X at the end of 24 years. Calculate X.

Problem 12.18
A loan of 12 is to be repaid with payments of 10 at the end of 3 years and 5 at the end of 6 years. Calculate the simple discount rate that is being charged on the loan.

Problem 12.19
A deposit of K^3 at time $t = 0$ accumulates to 8000 after $3n$ years using an annual effective discount rate of d. Using the same annual effective discount rate, a deposit of K^2 at time $t = 0$ accumulates to X after $2n$ years. Calculate X.

Problem 12.20
An investor puts 1000 into Fund X and 1000 into Fund Y. Fund Y earns compound interest at the annual rate of $j > 0$, and Fund X earns simple interest at the annual rate of $1.10j$. At the end of 2 years, the amount in Fund Y is 30 more than the amount in Fund X.
After 5 years, the amount in Fund Y exceeds the amount in Fund X by E. Determine E.

Problem 12.21
You are given two loans, with each loan to be repaid by a single payment in the future. Each payment includes both principal and interest. The first loan is repaid by a 3,000 payment at the

end of four years. The interest is accrued at 10% per annum compounded semi-annually. The second loan is repaid by a 4,000 payment at the end of five years. The interest is accrued at 8% per annum compounded semi-annually. These two loans are to be consolidated. The consolidated loan is to be repaid by two equal installments of X, with interest at 12% per annum compounded semi-annually. The first payment is due immediately and the second payment is due one year from now. Calculate X.

Problem 12.22
An investment fund accrues interest with force of interest

$$\delta_t = \frac{K}{1+(1-t)K}$$

for $0 \le t \le 1$. At time zero, there is 100,000 in the fund. At time one there is 110,000 in the fund. The only two transactions during the year are a deposit of 15,000 at time 0.25 and a withdrawal of 20,000 at time 0.75. Calculate K.

Problem 12.23
Paul borrows 1000 at an annual interest rate of 12%. He repays the loan in full by making the following payments:
(1) 500 at the end of 4 months
(2) 200 at the end of 14 months
(3) R at the end of 18 months
Calculate R.

13 Solving for the Unknown Interest Rate

In this section, it is the rate of interest i that is unknown in an equation of value. We discuss four ways for detrmining i.

Method 1: Direct Method
When a single payment is involved, the method that works best is to solve for i directly in the equation of value (using exponential and logarithmic functions). In this situation, the equation of value takes either the form $A = P(1+i)^n$ or $A = Pe^{\delta n}$. In the first case, the interest is found by using the exponential function obtaining

$$i = \left(\frac{A}{P}\right)^{\frac{1}{n}} - 1.$$

In the second case, the interest is found using the logarithmic function obtaining

$$\delta = \frac{1}{n}\ln\left(\frac{A}{P}\right).$$

Example 13.1
At what interest rate convertible semiannually would \$500 accumulate to \$800 in 4 years?

Solution.
Let $j = \frac{i^{(2)}}{2}$ be the effective rate per six months. We are given that $500(1+j)^8 = 800$. Solving for j and using a calculator we find $j = \left(\frac{8}{5}\right)^{\frac{1}{8}} - 1 \approx 0.06051$. Thus, $i^{(2)} = 2j = 0.121 = 12.1\%$ ■

Method 2: Analytical Method
When multiple payments are involved, the equation of value takes the form $f(i) = 0$. In this case, the problem is the classical problem of determining the nonnegative solutions to the equation $f(i) = 0$ with $f(i)$ being a differentiable function. If $f(i)$ is a polynomial then the problem reduces to solving polynomial equations. Some known algebraic methods can be used such as the rational zero test, or the quadratic formula, etc.

Example 13.2
At what effective interest rate would the present value of \$1000 at the end of 3 years plus \$2,000 at the end of 6 years be equal to \$2,700?

Solution.
The equation of value at time $t = 0$ is $2000\nu^6 + 1000\nu^3 = 2700$. Dividing through by 100 we find

$20\nu^6 + 10\nu^3 = 27$. Using the quadratic formula and the fact that $\nu^3 > 0$ we obtain $\nu^3 = 0.9385$ from which we obtain $\nu \approx 0.9791$. Hence, $i = \frac{1}{0.9791} - 1 \approx 0.0213 = 2.13\%$ ■

Method 3: Linear Interpolation
If the equation $f(i) = 0$ can not be solved known algebraic methods, then approximation methods can be used such as linear interpolation. We next introduce this concept.
Suppose that we know the values of a function $f(x)$ at distinct points x_1 and at x_2, and that $f(x_1) \neq f(x_2)$. If $|x_1 - x_2|$ is small, it may be reasonable to assume that the graph of f is approximately linear between x_1 and x_2. That is equivalent to assuming that

$$f(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$

for $x_1 < x < x_2$. If now we wish to determine an approximate value of x where $f(x)$ has a specific value y_0, we may use this equation for that purpose and find

$$x = x_1 + (x_2 - x_1)\frac{y_0 - f(x_1)}{f(x_2) - f(x_1)}.$$

In particular, if we wish to find a zero of f near the given points (i.e. $y_0 = 0$), and if $f(x_1) \neq f(x_2)$, a good approximation could be

$$x \approx x_1 - f(x_1)\frac{x_2 - x_1}{f(x_2) - f(x_1)}.$$

When we apply these formulas for points between two points x_1 and x_2, we speak of **linear interpolation.**
The following example illustrates the use of linear interpolation in determing the unknown rate.

Example 13.3
At what interest rate convertible semiannually would an investment of \$1,000 immediately and \$2,000 three years from now accumulate to \$5,000 ten years from now?

Solution.
The equation of value at time $t = 10$ years is

$$1000(1+j)^{20} + 2000(1+j)^{14} = 5000$$

where $j = \frac{i^{(2)}}{2}$. We will use linear interpolation to solve for j. For that purpose, we define

$$f(j) = 1000(1+j)^{20} + 2000(1+j)^{14} - 5000.$$

We want to find j such that $f(j) = 0$. By trial and error we see that $f(0.03) = -168.71$ and $f(0.035) = 227.17$. Since f is continuous, j is between these two values. Using linear interpolation we find

$$j \approx 0.03 + 168.71 \times \frac{0.005}{227.17 + 168.71} \approx 0.0321$$

and $f(0.0321) = -6.11$. Hence, $i^{(2)} = 2j = 0.0642 = 6.42\%$ ■

Method 4: Successive Iterations Methods

In order to achieve a higher level of accuracy than the one provided with linear interpolation, iteration methods come to mind. We will discuss two iteration methods: the bisection method and the Newton-Raphson method.

The **bisection method** is based on the fact that a differentiable function f that satisfies $f(\alpha)f(\beta) < 0$ must satisfy the equation $f(x_0) = 0$ for some x_0 between α and β.

The first step in the methods consists of finding two starting values $x_0 < x_1$ such that $f(x_0)f(x_1) < 0$. Usually, these values are found by trial and error. Next, bisect the interval by means of the midpoint $x_2 = \frac{x_0+x_1}{2}$. If $f(x_0)f(x_2) < 0$ then apply the bisection process to the interval $x_0 \le x \le x_2$. If $f(x_1)f(x_2) < 0$ then apply the bisection process to the interval $x_2 \le x \le x_1$. We continue the bisection process as many times as necessary to achieve the desired level of accuracy. We illustrate this method in the next example.

Example 13.4

At what interest rate convertible semiannually would an investment of \$1,000 immediately and \$2,000 three years from now accumulate to \$5,000 ten years from now? Use the bisection method.

Solution.

The equation of value at time $t = 10$ years is

$$1000(1 + j)^{20} + 2000(1 + j)^{14} = 5000$$

where $j = \frac{i^{(2)}}{2}$. Define

$$f(j) = 1000(1 + j)^{20} + 2000(1 + j)^{14} - 5000.$$

Descartes' rule of signs says that the maximum number of positive roots to a polynomial equation is equal to the number of sign changes in the coefficients of the polynomial. In our case, Descartes' rule asserts the existence of one positive root to the equation $f(j) = 0$.

Now, by trial and error we see that $f(0) = -2000$ and $f(0.1) = 9322.50$. Thus, j is between these two values. Let $j_2 = \frac{0+0.1}{2} = 0.05$. Then $f(0.05) = 1613.16$ so that $f(0)f(0.05) < 0$. Now bisect the interval $[0, 0.05]$ by means of the point $j_3 = 0.5(0 + 0.05) = 0.025$ and note that $f(j_3) = -535.44$. Continue this process to obtain the table

n	j_n	$f(j_n)$
0	0	-2000
1	0.1	9322.50
2	0.05	1613.16
3	0.025	-535.44
4	0.0375	436.75
5	0.03125	-72.55
6	0.034375	176.02
7	0.0328125	50.25
8	0.03203125	-11.52
9	0.032421875	19.27
10	0.0322265625	3.852
11	0.0321289063	-3.84
12	0.0321777344	0.005

Thus, $j \approx 0.032178$ accurate to six decimal places compared to 0.0321 found in Example 13.3. Hence, $i^2 = 2j = 0.06436 = 6.436\%$ ■

The problem with the bisection method is that the rate of convergence is slow. An iteration method with a faster rate of convergence is the Newton-Raphson method given by the iteration formula

$$j_{n+1} = j_n - \frac{f(j_n)}{f'(j_n)}.$$

This method is discussed in more details in Section 20.

Example 13.5
Rework Example 13.3 using the Newton-Raphson method.

Solution.
We have

$$f(j) = 1000(1+j)^{20} + 2000(1+j)^{14} - 5000.$$

and

$$f'(j) = 20000(1+j)^{19} + 28000(1+j)^{13}.$$

Thus,

$$j_{n+1} = j_n - \frac{(1+j_n)^{20} + 2(1+j_n)^{14} - 5}{20(1+j_n)^{19} + 28(1+j_n)^{13}}.$$

Let $j_0 = 0$. Then

$$\begin{aligned}
j_1 =& 0 - \frac{1+2-5}{20+28} = 0.0416666667\\
j_2 =& 0.0416666667 - \frac{1.0416666667^{20} + 2(1.0416666667)^{14} - 5}{20(1.0416666667)^{19} + 28(1.0416666667)^{13}} = 0.0328322051\\
j_3 =& 0.0328322051 - \frac{1.0328322051^{20} + 2(1.0328322051)^{14} - 5}{20(1.0328322051)^{19} + 28(1.0328322051)^{13}} = 0.0321809345\\
j_4 =& 0.0321809345 - \frac{1.0321809345^{20} + 2(1.0321809345)^{14} - 5}{20(1.0321809345)^{19} + 28(0.0321809345)^{13}} = 0.032177671
\end{aligned}$$

Thus, in four iterations we find $j = 0.032178$ and hence $i^{(2)} = 6.436\%$ ■

Practice Problems

Problem 13.1
Find the nominal rate of interest convertible semiannually at which the accumulated value of 1,000 at the end of 15 years is 3,000.

Problem 13.2
Find the exact effective rate of interest at which payments of 300 at the present, 200 at the end of one year, and 100 at the end of two years will accumulate to 700 at the end of two years.

Problem 13.3
The sum of the accumulated value of 1 at the end of three years at a certain effective rate of interest i, and the present value of 1 to be paid at the end of three years at an effective rate of discount numerically equal to i is 2.0096. Find the rate i.

Problem 13.4 ‡
David can receive one of the following two payment streams:
(i) 100 at time 0, 200 at time n, and 300 at time $2n$
(ii) 600 at time 10
At an annual effective interest rate of i, the present values of the two streams are equal.
Given $\nu^n = 0.75941$, determine i.

Problem 13.5 ‡
Mike receives cash flows of 100 today, 200 in one year, and 100 in two years. The present value of these cash flows is 364.46 at an annual effective rate of interest i. Calculate i.

Problem 13.6 ‡
At a nominal interest rate of i convertible semi-annually, an investment of 1,000 immediately and 1,500 at the end of the first year will accumulate to 2,600 at the end of the second year. Calculate i.

Problem 13.7 ‡
Eric deposits 100 into a savings account at time 0, which pays interest at a nominal rate of i, compounded semiannually.
Mike deposits 200 into a different savings account at time 0, which pays simple interest at an annual rate of i.
Eric and Mike earn the same amount of interest during the last 6 months of the 8th year.
Calculate i.

Problem 13.8 ‡
A deposit of 100 is made into a fund at time $t = 0$. The fund pays interest at a nominal rate of discount of d compounded quarterly for the first two years. Beginning at time $t = 2$, interest is credited at a force of interest $\delta_t = \frac{1}{t+1}$. At time $t = 5$, the accumulated value of the fund is 260. Calculate d.

Problem 13.9 ‡
A customer is offered an investment where interest is calculated according to the following force of interest:

$$\delta_t = \begin{cases} 0.02t & 0 \le t \le 3 \\ 0.045 & 3 < t. \end{cases}$$

The customer invests 1,000 at time $t = 0$. What nominal rate of interest, compounded quarterly, is earned over the first four-year period?

Problem 13.10
A manufacturer sells a product to a retailer who has the option of paying 30% below the retail price immediately, or 25% below the retail price in six months. Find the annual effective rate of interest at which the retailer would be indifferent between the two options.

Problem 13.11
An investor deposits 10,000 in a bank. During the first year, the bank credits an annual effective rate of interest i. During the second year the bank credits an annual effective interest $i - 0.05$. At the end of two years the account balance is 12,093.75. What would the account balance have been at the end of three years if the annual effective rate of interest were $i + 0.09$ for each of the three years?

Problem 13.12
You are given $\delta_t = \frac{2}{t-1}$ for any $t \in [2, 10]$. For any one year interval between n and $n + 1$; with $2 \le n \le 9$; calculate the equivalent nominal rate of discount $d^{(2)}$.

Problem 13.13
It is known that an investment of \$1,000 will accumulate to \$1,825 at the end of 10 years. If it is assumed that the investment earns simple interest at rate i during the 1st year, $2i$ during the second year, $\cdots$, $10i$ during the 10th year, find i.

Problem 13.14
It is known that an amount of money will double itself in 10 years at a varying force of interest $\delta_t = kt$. Find k.

Problem 13.15
A fund pays 1 at time $t = 0$, 2 at time $t = 2n$ and 1 at time $t = 4n$. The present value of the payments is 3.61. Calculate $(1 + i)^n$.

Problem 13.16
The present value of 300 in 3 years plus 600 in 6 years is equal to 800. Calculate i.

Problem 13.17
A savings and loan association pays 7% effective on deposits at the end of each year. At the end of every three years a 2% bonus is paid on the balance at that time. Find the effective rate of interest earned by an investor if the money is left on deposit:
(a) Two years.
(b) Three years.
(c) Four years.

Problem 13.18
Consider the following equation of value $25j = 1 - (1 + j)^{-40}$. Use linear interpolation to estimate j.

Problem 13.19
Consider the following equation of value $0.0725[1 - (1 + 0.01j)^{-50}] = 0.01j$. Use linear interpolation to estimate j.

Problem 13.20
Two funds, X and Y, start with the same amount. You are given
(1) Fund X accumulates at a force of interest 5%
(2) Fund Y accumulates at a rate of interest j compounded semi-annually
(3) At the end of eight years, Fund X is 1.05 times as large as Fund Y.
Calculate j.

Problem 13.21
You are given a loan on which interest is charged over a 4 year period, as follows
(1) an effective rate of discount of 6% for the first year
(2) a nominal rate of discount of 5% compounded every 2 years for the second year
(3) a nominal rate of interest of 5% compounded semiannually for the third year
(4) a force of interest of 5% for the fourth year.
Calculate the annual effective rate of interest over the 4 year period.

Problem 13.22
An investor deposits 10 into a fund today and 20 fifteen years later. Interest is credited at a nominal discount rate of d compounded quarterly for the first 10 years, and at a nominal interest rate of 6% compounded semi-annually thereafter. The accumulated balance in the fund at the end of 30 years is 100. Calculate d.

14 Solving for Unknown Time

As pointed out in Section 12, if any three of the four basic quantities: principal(s), investment period length(s), interest rate(s), accumulated value(s) are given, then the fourth can be determined. In this section, we consider the situation in which the length of the investment period is the unknown.

Example 14.1
A single payment of \$3,938.31 will pay off a debt whose original repayment plan was \$1,000 due on January 1 of each of the next four years, beginning January 1, 2006. If the effective annual rate is 8%, on what date must the payment of \$3,938.31 be paid?

Solution.
The equation of value with comparison date January 1, 2006 is

$$1,000[1 + (1 + 0.08)^{-1} + (1 + 0.08)^{-2} + (1 + 0.08)^{-3}] = 3938.31(1 + 0.08)^{-t}.$$

Solving this equation for t we find

$$t = \frac{\ln\{1000(3938.31)^{-1}[1 + (1 + 0.08)^{-1} + (1 + 0.08)^{-2} + (1 + 0.08)^{-3}]\}}{\ln 1.08} \approx 1.25.$$

Therefore, the single payment of 3938.31 must be paid 1.25 years after January 1, 2006, or on March 31, 2007 ■

In the above example, the answer was obtained by using a pocket calculator with exponential and logarithmic functions. An alternative approach to using calculator is the use of linear interpolation in the interest tables as illustrated in the next example.

Example 14.2
Find the length of time necessary for \$1000 to accumulate to \$1500 if invested at 6% per year compounded semiannually:
(a) by a direct method, i.e. using a calculator;
(b) by interpolating in the interest tables

Solution.
Let t be the comparison date. Then we must have $1000(1.03)^{2t} = 1000$ or $(1.03)^{2t} = 1.5$.
(a) Using a pocket calculator we find

$$t = \frac{1}{2}\frac{\ln 1.5}{\ln 1.03} = 6.859 \text{ years}.$$

(b) From interest tables, $(1.03)^{13} = 1.46853$ and $(1.03)^{14} = 1.51259$. Thus, $13 < 2t < 14$. Performing a linear interpolation we find

$$2t = 13 + \frac{1.5 - 1.46853}{1.51259 - 1.46853} = 13.714.$$

Thus, $t = 6.857$ years ■

The Rule of 72

Next, we consider the following problem: Given a particular rate of compound interest, how long will it take an investment to double?

Starting with 1 invested at an annually compounded interest i, let n be the time needed for the accumulated value to become 2. That is,

$$(1+i)^n = 2.$$

Taking the natural logarithm of both sides we find

$$n \ln(1+i) = \ln 2$$

and solving for n we obtain

$$n = \frac{\ln 2}{\ln(1+i)}.$$

We would like to find an approximation that is pretty good for an interest rate of $i = 8\%$. For that purpose, we start by writing the Taylor series approximation of $\frac{x}{\ln(1+x)}$ around $x = 0.08$ to obtain

$$\begin{aligned}\frac{x}{\ln(1+x)} =& 1.039486977 + 0.4874162023(x - 0.08) - 0.07425872903(x - 0.08)^2 + \cdots \\ \approx& 1.039486977\end{aligned}$$

for x very "close" to 0.08. Thus, the time necessary for the original principal to double is given by

$$t = \frac{\ln 2}{\ln(1+i)} = \frac{\ln 2}{i} \cdot \frac{i}{\ln(1+i)} \approx \frac{\ln 2}{i}(1.039486977) \approx \frac{0.72}{i}.$$

This is called the **rule of 72,** since t can be written in the form

$$t = \frac{72}{100i}.$$

Here's a table that shows the actual number of years required to double your money based on different interest rates, along with the number that the rule of 72 gives you.

% Rate	Actual	Rule 72
1	69.66	72
2	35.00	36
3	23.45	24
4	17.67	18
5	14.21	14.4
6	11.90	12
7	10.24	10.29
8	9.01	9
9	8.04	8
10	7.27	7.2
···	···	···
15	4.96	4.8
20	3.80	3.6
25	3.11	2.88
30	2.64	2.4
40	2.06	1.8
50	1.71	1.44
75	1.24	0.96
100	1.00	0.72

Example 14.3
Suppose \$2,000 is invested at a rate of 7% compounded semiannually.
(a) Find the length of time for the investment to double by using the exact formula.
(b) Find the length of time for the investment to double by using the rule of 72.

Solution.
(a) By using the exact formula for compound interest we find

$$2000(1+0.035)^{2t} = 4000.$$

Solving for t we find $t \approx 10.074$ years.
(b) Using the Rule of 72 we find

$$t = \frac{72}{7} \approx 10.286 \blacksquare$$

The Method of Equated Time
We consider the following problem: Suppose amounts $s_1, s_2, \cdots, s_n$ are to be paid at respective times $t_1, t_2, \cdots, t_n$ with annual interest rate i. What would be the time t when one single payment

of $s_1 + s_2 + \cdots + s_n$ would be equivalent to the individual payments made separately? A time diagram for this situation is shown in Figure 14.1

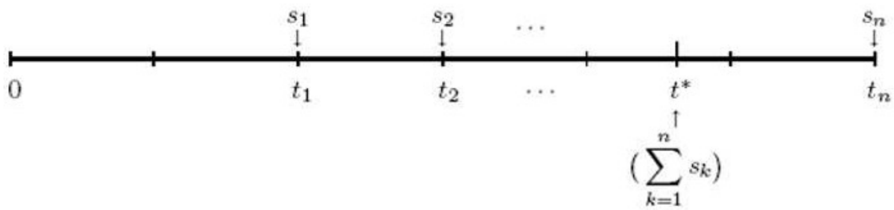


Figure 14.1

In this case, the equation of value at $t = 0$ is

$$(s_1 + s_2 + \cdots + s_n)\nu^{t^*} = s_1\nu^{t_1} + s_2\nu^{t_2} + \cdots + s_n\nu^{t_n}. \tag{14.1}$$

Taking the natural logarithm of both sides we find

$$t^* \ln \nu + \ln (s_1 + s_2 + \cdots + s_n) = \ln (s_1\nu^{t_1} + s_2\nu^{t_2} + \cdots + s_n\nu^{t_n}).$$

Solving for t^* and recalling that $\ln \nu = -\delta$ we find

$$\begin{aligned} t^* =& -\frac{1}{\delta} \ln \left(\frac{s_1\nu^{t_1} + s_2\nu^{t_2} + \cdots + s_n\nu^{t_n}}{s_1 + s_2 + \cdots + s_n} \right) \\ =& \frac{1}{\delta} \left[\ln \left(\sum_{k=1}^{n} s_k \right) - \ln \left(\sum_{k=1}^{n} s_k\nu^{t_k} \right) \right] \end{aligned}$$

In order to provide an approximation to t^* we recall the binomial series expansion

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots$$

where $\alpha \in \mathbb{R}$ and $-1 < x < 1$. Using this series expansion we can write

$$\nu^{t^*} = (1+i)^{-t^*} = 1 + \frac{(-t^*)}{1!}i + \frac{(-t^*)(-t^*-1)}{2!}i^2 + \cdots \approx 1 - it^*.$$

Similarly, we have

$$\nu^{t_k} = (1+i)^{-t_k} = 1 + \frac{(-t_k)}{1!}i + \frac{(-t_k)(-t_k-1)}{2!}i^2 + \cdots \approx 1 - it_k.$$

Substituting these approximations in the equation

$$\nu^{t^*} = \frac{s_1\nu^{t_1} + s_2\nu^{t_2} + \cdots + s_n\nu^{t_n}}{s_1 + s_2 + \cdots + s_n}$$

we find

$$1 - it^* \approx \frac{s_1(1 - it_1) + s_2(1 - it_2) + \cdots + s_n(1 - it_n)}{s_1 + s_2 + \cdots + s_n}$$

which can be written in the form

$$1 - it^* \approx 1 - i\frac{s_1t_1 + s_2t_2 + \cdots + s_nt_n}{s_1 + s_2 + \cdots + s_n}.$$

Solving for t^* we obtain the approximation

$$t^* \approx \frac{s_1t_1 + s_2t_2 + \cdots + s_nt_n}{s_1 + s_2 + \cdots + s_n} = \bar{t}.$$

Estimating t^* with $\bar{t}$ is known as the **method of equated time.** Note that the formula for $\bar{t}$ is similar to the center of mass that is covered in a calculus course where the weights are replaced by the various amounts paid.

Example 14.4
Payments of \$100, \$200, and \$500 are due at the ends of years 2,3, and 8, respectively. Assuming an effective rate of interest of 5% per year, find the point in time at which a payment of \$800 would be equivalent using the method of equated time.

Solution.
Using the method of equating time we find

$$\bar{t} = \frac{100 \cdot 2 + 200 \cdot 3 + 500 \cdot 8}{100 + 200 + 500} = 6 \text{ years} \blacksquare$$

With the approximation $\bar{t}$, the estimated present value for the equivalent single payment is

$$PV = (s_1 + s_2 + \cdots + s_n)\nu^{\bar{t}}.$$

The following result shows that $\bar{t}$ is an overestimate of t. Also, the result shows that the true present value exceeds the present value given by the method of equated time.

Theorem 14.1
With $s_1, s_2, \cdots, s_n, t_1, t_2, \cdots, t_n, \bar{t}$, and t as defined above we have

$$s_1\nu^{t_1} + s_2\nu^{t_2} + \cdots + s_n\nu^{t_n} > (s_1 + s_2 + \cdots + s_n)\nu^{\bar{t}}.$$

Hence, $\bar{t} > t$.

Proof.
Consider s_1 payments each equal to ν^{t_1}, s_2 payments each equal to $\nu^{t_2}, \cdots$, s_n payments each equal to ν^{t_n}. The arithmetic mean of these payments is

$$\frac{s_1\nu^{t_1} + s_2\nu^{t_2} + \cdots + s_n\nu^{t_n}}{s_1 + s_2 + \cdots + s_n}$$

and the geometric mean of these payments is

$$\left(\nu^{s_1t_1}\nu^{s_2t_2}\cdots\nu^{s_nt_n}\right)^{\frac{1}{s_1+s_2+\cdots+s_n}} = \nu^{\frac{s_1t_1+s_2t_2+\cdots+s_nt_n}{s_1+s_2+\cdots+s_n}} = \nu^{\bar{t}}.$$

In Problem 14.16 we show that the arithmetic mean is always greater than the geometric mean. Hence,

$$\frac{s_1\nu^{t_1} + s_2\nu^{t_2} + \cdots + s_n\nu^{t_n}}{s_1 + s_2 + \cdots + s_n} > \nu^{\bar{t}}$$

or

$$s_1\nu^{t_1} + s_2\nu^{t_2} + \cdots + s_n\nu^{t_n} > (s_1 + s_2 + \cdots + s_n)\nu^{\bar{t}}.$$

The left-hand side is the true present value and the right-hand side is the estimated present value. Note that this last equation implies that $\nu^t > \nu^{\bar{t}}$ by (14.1). Thus, $\nu^{t-\bar{t}} > 1$ and since $\nu < 1$ we must have $\bar{t} > t$ ■

Remark 14.1
As seen above, the solution by the method of equated time is relatively easy to compute, and is generally fairly accurate. Also, an interesting thing to notice is that the interest rate plays no role in the computations when the method of equated time is used.

Example 14.5
A loan is negotiated with the lender agreeing to accept \$1,000 after 10 years, \$2,000 after 20 years, and \$3,000 after 30 years in full repayments. The borrower wishes to liquidate the loan with a single \$6,000 payment. Let T_1 represent the time of the \$6,000 payment calculated by an equation of value. Let T_2 represent the time determined by the method of equated time. If $i = 0.01$, find $T_2 - T_1$ to the nearest penny.

Solution.
The equation of value at time $t = 0$ is

$$1,000\nu^{10} + 2,000\nu^{20} + 3,000\nu^{30} = 6,000\nu^{T_1}.$$

Solving we find

$$T_1 = \frac{\ln\left(\nu^{10} + 2\nu^{20} + 3\nu^{30}\right) - \ln 6}{-\ln 1.01} = 23.05.$$

Now, using the method of equated time we find

$$T_2 = \frac{1,000 \times 10 + 2,000 \times 20 + 3,000 \times 30}{6,000} = 23.33.$$

Thus, $T_2 - T_1 = 23.33 - 23.05 = 0.28$ ■

Practice Problems

Problem 14.1
The present value of a payment of \$5,000 to be made in t years is equal to the present value of a payment of \$7,100 to be made in $2t$ years. If $i = 7.5\%$ find t.

Problem 14.2
The present value of two payments of 100 each to be made at the end of n years and $2n$ years is 100. If $i = 0.08$, find n.

Problem 14.3
You are asked to develop a rule of n to approximate how long it takes money to triple. Find n.

Problem 14.4 ‡
Joe deposits 10 today and another 30 in five years into a fund paying simple interest of 11% per year.
Tina will make the same two deposits, but the 10 will be deposited n years from today and the 30 will be deposited $2n$ years from today. Tina's deposits earn an annual effective rate of 9.15% .
At the end of 10 years, the accumulated amount of Tina's deposits equals the accumulated amount of Joe's deposits. Calculate n.

Problem 14.5
Find how long \$1,000 should be left to accumulate at 6% effective in order that it will amount to twice the accumulated value of another 1,000 deposited at the same time at 4% effective.

Problem 14.6
If an investment will be doubled in 8 years at a force of interest δ, in how many years will an investment be tripled at a nominal rate of interest numerically equal to δ and convertible once every three years?

Problem 14.7
Amounts \$500, \$800, and \$1,000 are to be paid at respective times 3 years from today, 5 years from today, and 11 years from today. The effective rate of interest is 4.5% per annum.
(a) Use the method of equated time to find the number of years t from today when one single payment of \$500 + \$800 + \$1000 = \$2,300 would be equivalent to the individual payments made separately.
(b) Find the exact number of years t from today when one single payment of \$500 + \$800 + \$1000 = \$2,300 would be equivalent to the individual payments made separately.

Problem 14.8
A payment of n is made at the end of n years, $2n$ at the end of $2n$ years, $\cdots$, n^2 at the end of n^2 years. Find the value of t by the method of equated time.

Problem 14.9
Fund A accumulates at a rate of 12% convertible monthly. Fund B accumulates with a force of interest $\delta_t = \frac{t}{6}$. At time $t = 0$ equal deposits are made in each fund. Find the next time that the two funds are equal.

Problem 14.10
A loan requires the borrower to repay \$1,000 after 1 year, 2000 after 2 years, \$3,000 after 3 years, and \$4,000 after 4 years. At what time could the borrower make a single payment of \$10,000 according to the method of equated time?

Problem 14.11
A series of payments of 100 are made with the first payment made immediately. The second payment is made at the end of year 2. The third payment is made at the end of year 4, etc with the last payment being made at the end of year 12.
Calculate t using the method of equated time.

Problem 14.12
A series of payments are made at the end of each year for 50 years. The amount of the payment at the end of year n is n. Calculate t using the method of equated time.

Problem 14.13
A fund earns interest at a nominal rate of interest of 12% compounded monthly. If \$1,000 is invested in the fund, it will grow to be \$5,320.97 after n years. Calculate n.

Problem 14.14
A fund earns interest at a force of interest of $\delta_t = 0.05t$. Lauren invests \$2,000 at time $t = 0$. After n years, Lauren has \$4,919.20 . Calculate n.

Problem 14.15
A fund earns interest at a force of interest of $\delta_t = 0.01t^2$. Calculate the amount of time until the fund triples.

Problem 14.16
Consider the positive numbers $a_1, a_2, \cdots, a_n$ where n is a positive integer. The arithmetic mean is defined by

$$\frac{a_1 + a_2 + \cdots + a_n}{n}$$

whereas the geometric mean is defined by

$$[a_1 \cdot a_2 \cdots a_n]^{\frac{1}{n}}.$$

Show that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq [a_1 \cdot a_2 \cdots a_n]^{\frac{1}{n}}.$$

Hint: Use the fact that $e^x \geq 1 + x$ for all x. This implies $e^{\frac{a_i}{\mu}-1} \geq \frac{a_i}{\mu}, 1 \leq i \leq n$ where μ is the arithmetic mean.

Problem 14.17

A fund starts with a zero balance at time zero. The fund accumulates with a varying force of interest $\delta_t = \frac{2t}{t^2+1}$, for $t \geq 0$. A deposit of 100,000 is made at time 2. Calculate the number of years from the time of deposit for the fund to double.

Problem 14.18

At time $t = 0$ Billy puts 625 into an account paying 6% simple interest. At the end of year 2, George puts 400 into an account paying interest at a force of interest $\delta_t = \frac{1}{6+t}$, for $t \geq 2$. If both accounts continue to earn interest indefinitely at the levels given above, the amounts in both accounts will be equal at the end of year n. Calculate n.

Problem 14.19

Payment of 300, 500, and 700 are made at the end of years five, six, and eight, respectively. Interest is accumulated at an annual effective interest rate of 4%. You are to find the point in time at which a single payment of 1500 is equivalent to the above series of payments. You are given:
(i) X is the point in time calculated using the method of equated time.
(ii) Y is the exact point in time.
Calculate $X + Y$.

Problem 14.20

The present value of a payment of 1004 at the end of T months is equal to the present value of 314 after 1 month, 271 after 18 months, and 419 after 24 months. The effective annual interest rate is 5%. Calculate T to the nearest integer.

Problem 14.21

Using the method of equated time, a payment of 400 at time $t = 2$ plus a payment of X at time $t = 5$ is equivalent to a payment of $400 + X$ at time $t = 3.125$.
At an annual effective interest rate of 10%, the above two payments are equivalent to a payment of $400 + X$ at time $t = k$ using the exact method. Calculate k.

Problem 14.22

Investor A deposits 100 into an account that earns simple interest at a rate of 10% per annum.
Investor B makes two deposits of 50 each into an account that earns compound interest at an annual effective rate of 10%.
$B's$ first deposit occurs n years after $A's$ deposit.
$B's$ second deposit occurs $2n$ years after $A's$ deposit.
The balances in the two accounts are equal 10 years after Investor $A's$ deposit.
What is the value of n?

INTEREST TABLES AT 0.5%

Constants	
Function	Value
i	.005000
$i^{(2)}$	.004994
$i^{(4)}$	.004991
$i^{(12)}$	.004989
δ	.004988
d	.004975
$d^{(2)}$	.004981
$d^{(4)}$	.004984
$d^{(12)}$	.004987
δ	.004988
v	.995025
$v^{1/2}$	.997509
$v^{1/4}$	.998754
$v^{1/12}$	.999584
$1+i$	1.005000
$(1+i)^{1/2}$	1.002497
$(1+i)^{1/4}$	1.001248
$(1+i)^{1/12}$	1.000416
$i/i^{(2)}$	1.001248
$i/i^{(4)}$	1.001873
$i/i^{(12)}$	1.002290
i/δ	1.002498
$i/d^{(2)}$	1.003748
$i/d^{(4)}$	1.003123
$i/d^{(12)}$	1.002706
i/δ	1.002498

n	v^n	$(1+i)^n$	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$1/s_{\overline{n}\vert}$
1	.99502	1.00500	.9950	1.0000	1.000000
2	.99007	1.01003	1.9851	2.0050	.498753
3	.98515	1.01508	2.9702	3.0150	.331672
4	.98025	1.02015	3.9505	4.0301	.248133
5	.97537	1.02525	4.9259	5.0503	.198010
6	.97052	1.03038	5.8964	6.0755	.164595
7	.96569	1.03553	6.8621	7.1059	.140729
8	.96089	1.04071	7.8230	8.1414	.122829
9	.95610	1.04591	8.7791	9.1821	.108907
10	.95135	1.05114	9.7304	10.2280	.097771
11	.94661	1.05640	10.6770	11.2792	.088659
12	.94191	1.06168	11.6189	12.3356	.081066
13	.93722	1.06699	12.5562	13.3972	.074642
14	.93256	1.07232	13.4887	14.4642	.069136
15	.92792	1.07768	14.4166	15.5365	.064364
16	.92330	1.08307	15.3399	16.6142	.060189
17	.91871	1.08849	16.2586	17.6973	.056506
18	.91414	1.09393	17.1728	18.7858	.053232
19	.90959	1.09940	18.0824	19.8797	.050303
20	.90506	1.10490	18.9874	20.9791	.047666
21	.90056	1.11042	19.8880	22.0840	.045282
22	.89608	1.11597	20.7841	23.1944	.043114
23	.89162	1.12155	21.6757	24.3104	.041135
24	.88719	1.12716	22.5629	25.4320	.039321
25	.88277	1.13280	23.4456	26.5591	.037652
26	.87838	1.13846	24.3240	27.6919	.036112
27	.87401	1.14415	25.1980	28.8304	.034686
28	.86966	1.14987	26.0677	29.9745	.033362
29	.86533	1.15562	26.9330	31.1244	.032129
30	.86103	1.16140	27.7941	32.2800	.030979
31	.85675	1.16721	28.6508	33.4414	.029903
32	.85248	1.17304	29.5033	34.6086	.028895
33	.84824	1.17891	30.3515	35.7817	.027947
34	.84402	1.18480	31.1955	36.9606	.027056
35	.83982	1.19073	32.0354	38.1454	.026215
36	.83564	1.19668	32.8710	39.3361	.025422
37	.83149	1.20266	33.7025	40.5328	.024671
38	.82735	1.20868	34.5299	41.7354	.023960
39	.82323	1.21472	35.3531	42.9441	.023286
40	.81914	1.22079	36.1722	44.1588	.022646
41	.81506	1.22690	36.9873	45.3796	.022036
42	.81101	1.23303	37.7983	46.6065	.021456
43	.80697	1.23920	38.6053	47.8396	.020903
44	.80296	1.24539	39.4082	49.0788	.020375
45	.79896	1.25162	40.2072	50.3242	.019871
46	.79499	1.25788	41.0022	51.5758	.019389
47	.79103	1.26417	41.7932	52.8337	.018927
48	.78710	1.27049	42.5803	54.0978	.018485
49	.78318	1.27684	43.3635	55.3683	.018061
50	.77929	1.28323	44.1428	56.6452	.017654

INTEREST TABLES AT 1%

Constants	
Function	Value
i	.010000
$i^{(2)}$	.009975
$i^{(4)}$	.009963
$i^{(12)}$	.009954
δ	.009950
d	.009901
$d^{(2)}$	.009926
$d^{(4)}$	.009938
$d^{(12)}$	.009946
δ	.009950
v	.990099
$v^{1/2}$	.995037
$v^{1/4}$	.997516
$v^{1/12}$	.999171
$1+i$	1.010000
$(1+i)^{1/2}$	1.004988
$(1+i)^{1/4}$	1.002491
$(1+i)^{1/12}$	1.000830
$i/i^{(2)}$	1.002494
$i/i^{(4)}$	1.003742
$i/i^{(12)}$	1.004575
i/δ	1.004992
$i/d^{(2)}$	1.007494
$i/d^{(4)}$	1.006242
$i/d^{(12)}$	1.005408
i/δ	1.004992

n	v^n	$(1+i)^n$	$a_{\overline{n}\rvert}$	$s_{\overline{n}\rvert}$	$1/s_{\overline{n}\rvert}$
1	.99010	1.01000	.9901	1.0000	1.000000
2	.98030	1.02010	1.9704	2.0100	.497512
3	.97059	1.03030	2.9410	3.0301	.330022
4	.96098	1.04060	3.9020	4.0604	.246281
5	.95147	1.05101	4.8534	5.1010	.196040
6	.94205	1.06152	5.7955	6.1520	.162548
7	.93272	1.07214	6.7282	7.2135	.138628
8	.92348	1.08286	7.6517	8.2857	.120690
9	.91434	1.09369	8.5660	9.3685	.106740
10	.90529	1.10462	9.4713	10.4622	.095582
11	.89632	1.11567	10.3676	11.5668	.086454
12	.88745	1.12683	11.2551	12.6825	.078849
13	.87866	1.13809	12.1337	13.8093	.072415
14	.86996	1.14947	13.0037	14.9474	.066901
15	.86135	1.16097	13.8651	16.0969	.062124
16	.85282	1.17258	14.7179	17.2579	.057945
17	.84438	1.18430	15.5623	18.4304	.054258
18	.83602	1.19615	16.3983	19.6147	.050982
19	.82774	1.20811	17.2260	20.8109	.048052
20	.81954	1.22019	18.0456	22.0190	.045415
21	.81143	1.23239	18.8570	23.2392	.043031
22	.80340	1.24472	19.6604	24.4716	.040864
23	.79544	1.25716	20.4558	25.7163	.038886
24	.78757	1.26973	21.2434	26.9735	.037073
25	.77977	1.28243	22.0232	28.2432	.035407
26	.77205	1.29526	22.7952	29.5256	.033869
27	.76440	1.30821	23.5596	30.8209	.032446
28	.75684	1.32129	24.3164	32.1291	.031124
29	.74934	1.33450	25.0658	33.4504	.029895
30	.74192	1.34785	25.8077	34.7849	.028748
31	.73458	1.36133	26.5423	36.1327	.027676
32	.72730	1.37494	27.2696	37.4941	.026671
33	.72010	1.38869	27.9897	38.8690	.025727
34	.71297	1.40258	28.7027	40.2577	.024840
35	.70591	1.41660	29.4086	41.6603	.024004
36	.69892	1.43077	30.1075	43.0769	.023214
37	.69200	1.44508	30.7995	44.5076	.022468
38	.68515	1.45953	31.4847	45.9527	.021761
39	.67837	1.47412	32.1630	47.4123	.021092
40	.67165	1.48886	32.8347	48.8864	.020456
41	.66500	1.50375	33.4997	50.3752	.019851
42	.65842	1.51879	34.1581	51.8790	.019276
43	.65190	1.53398	34.8100	53.3978	.018727
44	.64545	1.54932	35.4555	54.9318	.018204
45	.63905	1.56481	36.0945	56.4811	.017705
46	.63273	1.58046	36.7272	58.0459	.017228
47	.62646	1.59626	37.3537	59.6263	.016771
48	.62026	1.61223	37.9740	61.2226	.016334
49	.61412	1.62835	38.5881	62.8348	.015915
50	.60804	1.64463	39.1961	64.4632	.015513

INTEREST TABLES AT 1.5%

Constants	
Function	Value
i	.015000
$i^{(2)}$	.014944
$i^{(4)}$	.014916
$i^{(12)}$	.014898
δ	.014889
d	.014778
$d^{(2)}$	.014833
$d^{(4)}$	.014861
$d^{(12)}$	.014879
δ	.014889
v	.985222
$v^{1/2}$	.992583
$v^{1/4}$	.996285
$v^{1/12}$	.998760
$1+i$	1.015000
$(1+i)^{1/2}$	1.007472
$(1+i)^{1/4}$	1.003729
$(1+i)^{1/12}$	1.001241
$i/i^{(2)}$	1.003736
$i/i^{(4)}$	1.005608
$i/i^{(12)}$	1.006857
i/δ	1.007481
$i/d^{(2)}$	1.011236
$i/d^{(4)}$	1.009358
$i/d^{(12)}$	1.008107
i/δ	1.007481

n	v^n	$(1+i)^n$	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$1/s_{\overline{n}\vert}$
1	.98522	1.01500	.9852	1.0000	1.000000
2	.97066	1.03023	1.9559	2.0150	.496278
3	.95632	1.04568	2.9122	3.0452	.328383
4	.94218	1.06136	3.8544	4.0909	.244445
5	.92826	1.07728	4.7826	5.1523	.194089
6	.91454	1.09344	5.6972	6.2296	.160525
7	.90103	1.10984	6.5982	7.3230	.136556
8	.88771	1.12649	7.4859	8.4328	.118584
9	.87459	1.14339	8.3605	9.5593	.104610
10	.86167	1.16054	9.2222	10.7027	.093434
11	.84893	1.17795	10.0711	11.8633	.084294
12	.83639	1.19562	10.9075	13.0412	.076680
13	.82403	1.21355	11.7315	14.2368	.070240
14	.81185	1.23176	12.5434	15.4504	.064723
15	.79985	1.25023	13.3432	16.6821	.059944
16	.78803	1.26899	14.1313	17.9324	.055765
17	.77639	1.28802	14.9076	19.2014	.052080
18	.76491	1.30734	15.6726	20.4894	.048806
19	.75361	1.32695	16.4262	21.7967	.045878
20	.74247	1.34686	17.1686	23.1237	.043246
21	.73150	1.36706	17.9001	24.4705	.040865
22	.72069	1.38756	18.6208	25.8376	.038703
23	.71004	1.40838	19.3309	27.2251	.036731
24	.69954	1.42950	20.0304	28.6335	.034924
25	.68921	1.45095	20.7196	30.0630	.033263
26	.67902	1.47271	21.3986	31.5140	.031732
27	.66899	1.49480	22.0676	32.9867	.030315
28	.65910	1.51722	22.7267	34.4815	.029001
29	.64936	1.53998	23.3761	35.9987	.027779
30	.63976	1.56308	24.0158	37.5387	.026639
31	.63031	1.58653	24.6461	39.1018	.025574
32	.62099	1.61032	25.2671	40.6883	.024577
33	.61182	1.63448	25.8790	42.2986	.023641
34	.60277	1.65900	26.4817	43.9331	.022762
35	.59387	1.68388	27.0756	45.5921	.021934
36	.58509	1.70914	27.6607	47.2760	.021152
37	.57644	1.73478	28.2371	48.9851	.020414
38	.56792	1.76080	28.8051	50.7199	.019716
39	.55953	1.78721	29.3646	52.4807	.019055
40	.55126	1.81402	29.9158	54.2679	.018427
41	.54312	1.84123	30.4590	56.0819	.017831
42	.53509	1.86885	30.9941	57.9231	.017264
43	.52718	1.89688	31.5212	59.7920	.016725
44	.51939	1.92533	32.0406	61.6889	.016210
45	.51171	1.95421	32.5523	63.6142	.015720
46	.50415	1.98353	33.0565	65.5684	.015251
47	.49670	2.01328	33.5532	67.5519	.014803
48	.48936	2.04348	34.0426	69.5652	.014375
49	.48213	2.07413	34.5247	71.6087	.013965
50	.47500	2.10524	34.9997	73.6828	.013572

INTEREST TABLES AT 2%

Constants	
Function	Value
i	.020000
$i^{(2)}$	.019901
$i^{(4)}$	.019852
$i^{(12)}$	.019819
δ	.019803
d	.019608
$d^{(2)}$	.019705
$d^{(4)}$	.019754
$d^{(12)}$	.019786
δ	.019803
v	.980392
$v^{1/2}$	.990148
$v^{1/4}$	.995062
$v^{1/12}$	.998351
$1+i$	1.020000
$(1+i)^{1/2}$	1.009950
$(1+i)^{1/4}$	1.004963
$(1+i)^{1/12}$	1.001652
$i/i^{(2)}$	1.004975
$i/i^{(4)}$	1.007469
$i/i^{(12)}$	1.009134
i/δ	1.009967
$i/d^{(2)}$	1.014975
$i/d^{(4)}$	1.012469
$i/d^{(12)}$	1.010801
i/δ	1.009967

n	v^n	$(1+i)^n$	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$1/s_{\overline{n}\vert}$
1	.98039	1.02000	.9804	1.0000	1.000000
2	.96117	1.04040	1.9416	2.0200	.495050
3	.94232	1.06121	2.8839	3.0604	.326755
4	.92385	1.08243	3.8077	4.1216	.242624
5	.90573	1.10408	4.7135	5.2040	.192158
6	.88797	1.12616	5.6014	6.3081	.158526
7	.87056	1.14869	6.4720	7.4343	.134512
8	.85349	1.17166	7.3255	8.5830	.116510
9	.83676	1.19509	8.1622	9.7546	.102515
10	.82035	1.21899	8.9826	10.9497	.091327
11	.80426	1.24337	9.7868	12.1687	.082178
12	.78849	1.26824	10.5753	13.4121	.074560
13	.77303	1.29361	11.3484	14.6803	.068118
14	.75788	1.31948	12.1062	15.9739	.062602
15	.74301	1.34587	12.8493	17.2934	.057825
16	.72845	1.37279	13.5777	18.6393	.053650
17	.71416	1.40024	14.2919	20.0121	.049970
18	.70016	1.42825	14.9920	21.4123	.046702
19	.68643	1.45681	15.6785	22.8406	.043782
20	.67297	1.48595	16.3514	24.2974	.041157
21	.65978	1.51567	17.0112	25.7833	.038785
22	.64684	1.54598	17.6580	27.2990	.036631
23	.63416	1.57690	18.2922	28.8450	.034668
24	.62172	1.60844	18.9139	30.4219	.032871
25	.60953	1.64061	19.5235	32.0303	.031220
26	.59758	1.67342	20.1210	33.6709	.029699
27	.58586	1.70689	20.7069	35.3443	.028293
28	.57437	1.74102	21.2813	37.0512	.026990
29	.56311	1.77584	21.8444	38.7922	.025778
30	.55207	1.81136	22.3965	40.5681	.024650
31	.54125	1.84759	22.9377	42.3794	.023596
32	.53063	1.88454	23.4683	44.2270	.022611
33	.52023	1.92223	23.9886	46.1116	.021687
34	.51003	1.96068	24.4986	48.0338	.020819
35	.50003	1.99989	24.9986	49.9945	.020002
36	.49022	2.03989	25.4888	51.9944	.019233
37	.48061	2.08069	25.9695	54.0343	.018507
38	.47119	2.12230	26.4406	56.1149	.017821
39	.46195	2.16474	26.9026	58.2372	.017171
40	.45289	2.20804	27.3555	60.4020	.016556
41	.44401	2.25220	27.7995	62.6100	.015972
42	.43530	2.29724	28.2348	64.8622	.015417
43	.42677	2.34319	28.6616	67.1595	.014890
44	.41840	2.39005	29.0800	69.5027	.014388
45	.41020	2.43785	29.4902	71.8927	.013910
46	.40215	2.48661	29.8923	74.3306	.013453
47	.39427	2.53634	30.2866	76.8172	.013018
48	.38654	2.58707	30.6731	79.3535	.012602
49	.37896	2.63881	31.0521	81.9406	.012204
50	.37153	2.69159	31.4236	84.5794	.011823

INTEREST TABLES AT 2.5%

Constants	
Function	Value
i	.025000
$i^{(2)}$	.024846
$i^{(4)}$	.024769
$i^{(12)}$	.024718
δ	.024693
d	.024390
$d^{(2)}$	.024541
$d^{(4)}$	.024617
$d^{(12)}$	.024667
δ	.024693
v	.975610
$v^{1/2}$	.987730
$v^{1/4}$	.993846
$v^{1/12}$	.997944
$1+i$	1.025000
$(1+i)^{1/2}$	1.012423
$(1+i)^{1/4}$	1.006192
$(1+i)^{1/12}$	1.002060
$i/i^{(2)}$	1.006211
$i/i^{(4)}$	1.009327
$i/i^{(12)}$	1.011407
i/δ	1.012449
$i/d^{(2)}$	1.018711
$i/d^{(4)}$	1.015577
$i/d^{(12)}$	1.013491
i/δ	1.012449

n	v^n	$(1+i)^n$	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$1/s_{\overline{n}\vert}$
1	.97561	1.02500	.9756	1.0000	1.000000
2	.95181	1.05063	1.9274	2.0250	.493827
3	.92860	1.07689	2.8560	3.0756	.325137
4	.90595	1.10381	3.7620	4.1525	.240818
5	.88385	1.13141	4.6458	5.2563	.190247
6	.86230	1.15969	5.5081	6.3877	.156550
7	.84127	1.18869	6.3494	7.5474	.132495
8	.82075	1.21840	7.1701	8.7361	.114467
9	.80073	1.24886	7.9709	9.9545	.100457
10	.78120	1.28008	8.7521	11.2034	.089259
11	.76214	1.31209	9.5142	12.4835	.080106
12	.74356	1.34489	10.2578	13.7956	.072487
13	.72542	1.37851	10.9832	15.1404	.066048
14	.70773	1.41297	11.6909	16.5190	.060537
15	.69047	1.44830	12.3814	17.9319	.055766
16	.67362	1.48451	13.0550	19.3802	.051599
17	.65720	1.52162	13.7122	20.8647	.047928
18	.64117	1.55966	14.3534	22.3863	.044670
19	.62553	1.59865	14.9789	23.9460	.041761
20	.61027	1.63862	15.5892	25.5447	.039147
21	.59539	1.67958	16.1845	27.1833	.036787
22	.58086	1.72157	16.7654	28.8629	.034647
23	.56670	1.76461	17.3321	30.5844	.032696
24	.55288	1.80873	17.8850	32.3490	.030913
25	.53939	1.85394	18.4244	34.1578	.029276
26	.52623	1.90029	18.9506	36.0117	.027769
27	.51340	1.94780	19.4640	37.9120	.026377
28	.50088	1.99650	19.9649	39.8598	.025088
29	.48866	2.04641	20.4535	41.8563	.023891
30	.47674	2.09757	20.9303	43.9027	.022778
31	.46511	2.15001	21.3954	46.0003	.021739
32	.45377	2.20376	21.8492	48.1503	.020768
33	.44270	2.25885	22.2919	50.3540	.019859
34	.43191	2.31532	22.7238	52.6129	.019007
35	.42137	2.37321	23.1452	54.9282	.018206
36	.41109	2.43254	23.5563	57.3014	.017452
37	.40107	2.49335	23.9573	59.7339	.016741
38	.39128	2.55568	24.3486	62.2273	.016070
39	.38174	2.61957	24.7303	64.7830	.015436
40	.37243	2.68506	25.1028	67.4026	.014836
41	.36335	2.75219	25.4661	70.0876	.014268
42	.35448	2.82100	25.8206	72.8398	.013729
43	.34584	2.89152	26.1664	75.6608	.013217
44	.33740	2.96381	26.5038	78.5523	.012730
45	.32917	3.03790	26.8330	81.5161	.012268
46	.32115	3.11385	27.1542	84.5540	.011827
47	.31331	3.19170	27.4675	87.6679	.011407
48	.30567	3.27149	27.7732	90.8596	.011006
49	.29822	3.35328	28.0714	94.1311	.010623
50	.29094	3.43711	28.3623	97.4843	.010258

INTEREST TABLES AT 3%

Constants	
Function	Value
i	.030000
$i^{(2)}$	.029778
$i^{(4)}$	.029668
$i^{(12)}$	.029595
δ	.029559
d	.029126
$d^{(2)}$	.029341
$d^{(4)}$	.029450
$d^{(12)}$	.029522
δ	.029559
v	.970874
$v^{1/2}$	.985329
$v^{1/4}$	.992638
$v^{1/12}$	.997540
$1+i$	1.030000
$(1+i)^{1/2}$	1.014889
$(1+i)^{1/4}$	1.007417
$(1+i)^{1/12}$	1.002466
$i/i^{(2)}$	1.007445
$i/i^{(4)}$	1.011181
$i/i^{(12)}$	1.013677
i/δ	1.014926
$i/d^{(2)}$	1.022445
$i/d^{(4)}$	1.018681
$i/d^{(12)}$	1.016177
i/δ	1.014926

n	v^n	$(1+i)^n$	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$1/s_{\overline{n}\vert}$
1	.97087	1.03000	.9709	1.0000	1.000000
2	.94260	1.06090	1.9135	2.0300	.492611
3	.91514	1.09273	2.8286	3.0909	.323530
4	.88849	1.12551	3.7171	4.1836	.239027
5	.86261	1.15927	4.5797	5.3091	.188355
6	.83748	1.19405	5.4172	6.4684	.154598
7	.81309	1.22987	6.2303	7.6625	.130506
8	.78941	1.26677	7.0197	8.8923	.112456
9	.76642	1.30477	7.7861	10.1591	.098434
10	.74409	1.34392	8.5302	11.4639	.087231
11	.72242	1.38423	9.2526	12.8078	.078077
12	.70138	1.42576	9.9540	14.1920	.070462
13	.68095	1.46853	10.6350	15.6178	.064030
14	.66112	1.51259	11.2961	17.0863	.058526
15	.64186	1.55797	11.9379	18.5989	.053767
16	.62317	1.60471	12.5611	20.1569	.049611
17	.60502	1.65285	13.1661	21.7616	.045953
18	.58739	1.70243	13.7535	23.4144	.042709
19	.57029	1.75351	14.3238	25.1169	.039814
20	.55368	1.80611	14.8775	26.8704	.037216
21	.53755	1.86029	15.4150	28.6765	.034872
22	.52189	1.91610	15.9369	30.5368	.032747
23	.50669	1.97359	16.4436	32.4529	.030814
24	.49193	2.03279	16.9355	34.4265	.029047
25	.47761	2.09378	17.4131	36.4593	.027428
26	.46369	2.15659	17.8768	38.5530	.025938
27	.45019	2.22129	18.3270	40.7096	.024564
28	.43708	2.28793	18.7641	42.9309	.023293
29	.42435	2.35657	19.1885	45.2189	.022115
30	.41199	2.42726	19.6004	47.5754	.021019
31	.39999	2.50008	20.0004	50.0027	.019999
32	.38834	2.57508	20.3888	52.5028	.019047
33	.37703	2.65234	20.7658	55.0778	.018156
34	.36604	2.73191	21.1318	57.7302	.017322
35	.35538	2.81386	21.4872	60.4621	.016539
36	.34503	2.89828	21.8323	63.2759	.015804
37	.33498	2.98523	22.1672	66.1742	.015112
38	.32523	3.07478	22.4925	69.1594	.014459
39	.31575	3.16703	22.8082	72.2342	.013844
40	.30656	3.26204	23.1148	75.4013	.013262
41	.29763	3.35990	23.4124	78.6633	.012712
42	.28896	3.46070	23.7014	82.0232	.012192
43	.28054	3.56452	23.9819	85.4839	.011698
44	.27237	3.67145	24.2543	89.0484	.011230
45	.26444	3.78160	24.5187	92.7199	.010785
46	.25674	3.89504	24.7754	96.5015	.010363
47	.24926	4.01190	25.0247	100.3965	.009961
48	.24200	4.13225	25.2667	104.4084	.009578
49	.23495	4.25622	25.5017	108.5406	.009213
50	.22811	4.38391	25.7298	112.7969	.008865

INTEREST TABLES AT 3.5%

Constants	
Function	Value
i	.035000
$i^{(2)}$	.034699
$i^{(4)}$	.034550
$i^{(12)}$	.034451
δ	.034401
d	.033816
$d^{(2)}$	.034107
$d^{(4)}$	.034254
$d^{(12)}$	.034352
δ	.034401
v	.966184
$v^{1/2}$	.982946
$v^{1/4}$	.991437
$v^{1/12}$	.997137
$1+i$	1.035000
$(1+i)^{1/2}$	1.017349
$(1+i)^{1/4}$	1.008637
$(1+i)^{1/12}$	1.002871
$i/i^{(2)}$	1.008675
$i/i^{(4)}$	1.013031
$i/i^{(12)}$	1.015942
i/δ	1.017400
$i/d^{(2)}$	1.026175
$i/d^{(4)}$	1.021781
$i/d^{(12)}$	1.018859
i/δ	1.017400

n	v^n	$(1+i)^n$	$a_{\overline{n}\mid}$	$s_{\overline{n}\mid}$	$1/s_{\overline{n}\mid}$
1	.96618	1.03500	.9662	1.0000	1.000000
2	.93351	1.07123	1.8997	2.0350	.491400
3	.90194	1.10872	2.8016	3.1062	.321934
4	.87144	1.14752	3.6731	4.2149	.237251
5	.84197	1.18769	4.5151	5.3625	.186481
6	.81350	1.22926	5.3286	6.5502	.152668
7	.78599	1.27228	6.1145	7.7794	.128544
8	.75941	1.31681	6.8740	9.0517	.110477
9	.73373	1.36290	7.6077	10.3685	.096446
10	.70892	1.41060	8.3166	11.7314	.085241
11	.68495	1.45997	9.0016	13.1420	.076092
12	.66178	1.51107	9.6633	14.6020	.068484
13	.63940	1.56396	10.3027	16.1130	.062062
14	.61778	1.61869	10.9205	17.6770	.056571
15	.59689	1.67535	11.5174	19.2957	.051825
16	.57671	1.73399	12.0941	20.9710	.047685
17	.55720	1.79468	12.6513	22.7050	.044043
18	.53836	1.85749	13.1897	24.4997	.040817
19	.52016	1.92250	13.7098	26.3572	.037940
20	.50257	1.98979	14.2124	28.2797	.035361
21	.48557	2.05943	14.6980	30.2695	.033037
22	.46915	2.13151	15.1671	32.3289	.030932
23	.45329	2.20611	15.6204	34.4604	.029019
24	.43796	2.28333	16.0584	36.6665	.027273
25	.42315	2.36324	16.4815	38.9499	.025674
26	.40884	2.44596	16.8904	41.3131	.024205
27	.39501	2.53157	17.2854	43.7591	.022852
28	.38165	2.62017	17.6670	46.2906	.021603
29	.36875	2.71188	18.0358	48.9108	.020445
30	.35628	2.80679	18.3920	51.6227	.019371
31	.34423	2.90503	18.7363	54.4295	.018372
32	.33259	3.00671	19.0689	57.3345	.017442
33	.32134	3.11194	19.3902	60.3412	.016572
34	.31048	3.22086	19.7007	63.4532	.015760
35	.29998	3.33359	20.0007	66.6740	.014998
36	.28983	3.45027	20.2905	70.0076	.014284
37	.28003	3.57103	20.5705	73.4579	.013613
38	.27056	3.69601	20.8411	77.0289	.012982
39	.26141	3.82537	21.1025	80.7249	.012388
40	.25257	3.95926	21.3551	84.5503	.011827
41	.24403	4.09783	21.5991	88.5095	.011298
42	.23578	4.24126	21.8349	92.6074	.010798
43	.22781	4.38970	22.0627	96.8486	.010325
44	.22010	4.54334	22.2828	101.2383	.009878
45	.21266	4.70236	22.4955	105.7817	.009453
46	.20547	4.86694	22.7009	110.4840	.009051
47	.19852	5.03728	22.8994	115.3510	.008669
48	.19181	5.21359	23.0912	120.3883	.008306
49	.18532	5.39606	23.2766	125.6018	.007962
50	.17905	5.58493	23.4556	130.9979	.007634

INTEREST TABLES AT 4%

Constants	
Function	Value
i	.040000
$i^{(2)}$	.039608
$i^{(4)}$	.039414
$i^{(12)}$	.039285
δ	.039221
d	.038462
$d^{(2)}$	.038839
$d^{(4)}$	.039029
$d^{(12)}$	.039157
δ	.039221
v	.961538
$v^{1/2}$	.980581
$v^{1/4}$	.990243
$v^{1/12}$	.996737
$1+i$	1.040000
$(1+i)^{1/2}$	1.019804
$(1+i)^{1/4}$	1.009853
$(1+i)^{1/12}$	1.003274
$i/i^{(2)}$	1.009902
$i/i^{(4)}$	1.014877
$i/i^{(12)}$	1.018204
i/δ	1.019869
$i/d^{(2)}$	1.029902
$i/d^{(4)}$	1.024877
$i/d^{(12)}$	1.021537
i/δ	1.019869

n	v^n	$(1+i)^n$	$a_{\overline{n}\rvert}$	$s_{\overline{n}\rvert}$	$1/s_{\overline{n}\rvert}$
1	.96154	1.04000	.9615	1.0000	1.000000
2	.92456	1.08160	1.8861	2.0400	.490196
3	.88900	1.12486	2.7751	3.1216	.320349
4	.85480	1.16986	3.6299	4.2465	.235490
5	.82193	1.21665	4.4518	5.4163	.184627
6	.79031	1.26532	5.2421	6.6330	.150762
7	.75992	1.31593	6.0021	7.8983	.126610
8	.73069	1.36857	6.7327	9.2142	.108528
9	.70259	1.42331	7.4353	10.5828	.094493
10	.67556	1.48024	8.1109	12.0061	.083291
11	.64958	1.53945	8.7605	13.4864	.074149
12	.62460	1.60103	9.3851	15.0258	.066552
13	.60057	1.66507	9.9856	16.6268	.060144
14	.57748	1.73168	10.5631	18.2919	.054669
15	.55526	1.80094	11.1184	20.0236	.049941
16	.53391	1.87298	11.6523	21.8245	.045820
17	.51337	1.94790	12.1657	23.6975	.042199
18	.49363	2.02582	12.6593	25.6454	.038993
19	.47464	2.10685	13.1339	27.6712	.036139
20	.45639	2.19112	13.5903	29.7781	.033582
21	.43883	2.27877	14.0292	31.9692	.031280
22	.42196	2.36992	14.4511	34.2480	.029199
23	.40573	2.46472	14.8568	36.6179	.027309
24	.39012	2.56330	15.2470	39.0826	.025587
25	.37512	2.66584	15.6221	41.6459	.024012
26	.36069	2.77247	15.9828	44.3117	.022567
27	.34682	2.88337	16.3296	47.0842	.021239
28	.33348	2.99870	16.6631	49.9676	.020013
29	.32065	3.11865	16.9837	52.9663	.018880
30	.30832	3.24340	17.2920	56.0849	.017830
31	.29646	3.37313	17.5885	59.3283	.016855
32	.28506	3.50806	17.8736	62.7015	.015949
33	.27409	3.64838	18.1476	66.2095	.015104
34	.26355	3.79432	18.4112	69.8579	.014315
35	.25342	3.94609	18.6646	73.6522	.013577
36	.24367	4.10393	18.9083	77.5983	.012887
37	.23430	4.26809	19.1426	81.7022	.012240
38	.22529	4.43881	19.3679	85.9703	.011632
39	.21662	4.61637	19.5845	90.4091	.011061
40	.20829	4.80102	19.7928	95.0255	.010523
41	.20028	4.99306	19.9931	99.8265	.010017
42	.19257	5.19278	20.1856	104.8196	.009540
43	.18517	5.40050	20.3708	110.0124	.009090
44	.17805	5.61652	20.5488	115.4129	.008665
45	.17120	5.84118	20.7200	121.0294	.008262
46	.16461	6.07482	20.8847	126.8706	.007882
47	.15828	6.31782	21.0429	132.9454	.007522
48	.15219	6.57053	21.1951	139.2632	.007181
49	.14634	6.83335	21.3415	145.8337	.006857
50	.14071	7.10668	21.4822	152.6671	.006550

INTEREST TABLES AT 4.5%

Constants	
Function	**Value**
i	.045000
$i^{(2)}$	.044505
$i^{(4)}$	.044260
$i^{(12)}$	.044098
δ	.044017
d	.043062
$d^{(2)}$	.043536
$d^{(4)}$	.043776
$d^{(12)}$	.043936
δ	.044017
v	.956938
$v^{1/2}$	.978232
$v^{1/4}$	.989056
$v^{1/12}$	.996339
$1+i$	1.045000
$(1+i)^{1/2}$	1.022252
$(1+i)^{1/4}$	1.011065
$(1+i)^{1/12}$	1.003675
$i/i^{(2)}$	1.011126
$i/i^{(4)}$	1.016720
$i/i^{(12)}$	1.020461
i/δ	1.022335
$i/d^{(2)}$	1.033626
$i/d^{(4)}$	1.027970
$i/d^{(12)}$	1.024211
i/δ	1.022335

n	v^n	$(1+i)^n$	$a_{\overline{n}\mid}$	$s_{\overline{n}\mid}$	$1/s_{\overline{n}\mid}$
1	.95694	1.04500	.9569	1.0000	1.000000
2	.91573	1.09203	1.8727	2.0450	.488998
3	.87630	1.14117	2.7490	3.1370	.318773
4	.83856	1.19252	3.5875	4.2782	.233744
5	.80245	1.24618	4.3900	5.4707	.182792
6	.76790	1.30226	5.1579	6.7169	.148878
7	.73483	1.36086	5.8927	8.0192	.124701
8	.70319	1.42210	6.5959	9.3800	.106610
9	.67290	1.48610	7.2688	10.8021	.092574
10	.64393	1.55297	7.9127	12.2882	.081379
11	.61620	1.62285	8.5289	13.8412	.072248
12	.58966	1.69588	9.1186	15.4640	.064666
13	.56427	1.77220	9.6829	17.1599	.058275
14	.53997	1.85194	10.2228	18.9321	.052820
15	.51672	1.93528	10.7395	20.7841	.048114
16	.49447	2.02237	11.2340	22.7193	.044015
17	.47318	2.11338	11.7072	24.7417	.040418
18	.45280	2.20848	12.1600	26.8551	.037237
19	.43330	2.30786	12.5933	29.0636	.034407
20	.41464	2.41171	13.0079	31.3714	.031876
21	.39679	2.52024	13.4047	33.7831	.029601
22	.37970	2.63365	13.7844	36.3034	.027546
23	.36335	2.75217	14.1478	38.9370	.025682
24	.34770	2.87601	14.4955	41.6892	.023987
25	.33273	3.00543	14.8282	44.5652	.022439
26	.31840	3.14068	15.1466	47.5706	.021021
27	.30469	3.28201	15.4513	50.7113	.019719
28	.29157	3.42970	15.7429	53.9933	.018521
29	.27902	3.58404	16.0219	57.4230	.017415
30	.26700	3.74532	16.2889	61.0071	.016392
31	.25550	3.91386	16.5444	64.7524	.015443
32	.24450	4.08998	16.7889	68.6662	.014563
33	.23397	4.27403	17.0229	72.7562	.013745
34	.22390	4.46636	17.2468	77.0303	.012982
35	.21425	4.66735	17.4610	81.4966	.012270
36	.20503	4.87738	17.6660	86.1640	.011606
37	.19620	5.09686	17.8622	91.0413	.010984
38	.18775	5.32622	18.0500	96.1382	.010402
39	.17967	5.56590	18.2297	101.4644	.009856
40	.17193	5.81636	18.4016	107.0303	.009343
41	.16453	6.07810	18.5661	112.8467	.008862
42	.15744	6.35162	18.7235	118.9248	.008409
43	.15066	6.63744	18.8742	125.2764	.007982
44	.14417	6.93612	19.0184	131.9138	.007581
45	.13796	7.24825	19.1563	138.8500	.007202
46	.13202	7.57442	19.2884	146.0982	.006845
47	.12634	7.91527	19.4147	153.6726	.006507
48	.12090	8.27146	19.5356	161.5879	.006189
49	.11569	8.64367	19.6513	169.8594	.005887
50	.11071	9.03264	19.7620	178.5030	.005602

INTEREST TABLES AT 5%

Constants	
Function	**Value**
i	.050000
$i^{(2)}$	.049390
$i^{(4)}$	.049089
$i^{(12)}$	.048889
δ	.048790
d	.047619
$d^{(2)}$	.048200
$d^{(4)}$	.048494
$d^{(12)}$	.048691
δ	.048790
v	.952381
$v^{1/2}$	.975900
$v^{1/4}$	.987877
$v^{1/12}$	.995942
$1+i$	1.050000
$(1+i)^{1/2}$	1.024695
$(1+i)^{1/4}$	1.012272
$(1+i)^{1/12}$	1.004074
$i/i^{(2)}$	1.012348
$i/i^{(4)}$	1.018559
$i/i^{(12)}$	1.022715
i/δ	1.024797
$i/d^{(2)}$	1.037348
$i/d^{(4)}$	1.031059
$i/d^{(12)}$	1.026881
i/δ	1.024797

n	v^n	$(1+i)^n$	$a_{\overline{n}\rvert}$	$s_{\overline{n}\rvert}$	$1/s_{\overline{n}\rvert}$
1	.95238	1.05000	.9524	1.0000	1.000000
2	.90703	1.10250	1.8594	2.0500	.487805
3	.86384	1.15763	2.7232	3.1525	.317209
4	.82270	1.21551	3.5460	4.3101	.232012
5	.78353	1.27628	4.3295	5.5256	.180975
6	.74622	1.34010	5.0757	6.8019	.147017
7	.71068	1.40710	5.7864	8.1420	.122820
8	.67684	1.47746	6.4632	9.5491	.104722
9	.64461	1.55133	7.1078	11.0266	.090690
10	.61391	1.62889	7.7217	12.5779	.079505
11	.58468	1.71034	8.3064	14.2068	.070389
12	.55684	1.79586	8.8633	15.9171	.062825
13	.53032	1.88565	9.3936	17.7130	.056456
14	.50507	1.97993	9.8986	19.5986	.051024
15	.48102	2.07893	10.3797	21.5786	.046342
16	.45811	2.18287	10.8378	23.6575	.042270
17	.43630	2.29202	11.2741	25.8404	.038699
18	.41552	2.40662	11.6896	28.1324	.035546
19	.39573	2.52695	12.0853	30.5390	.032745
20	.37689	2.65330	12.4622	33.0660	.030243
21	.35894	2.78596	12.8212	35.7193	.027996
22	.34185	2.92526	13.1630	38.5052	.025971
23	.32557	3.07152	13.4886	41.4305	.024137
24	.31007	3.22510	13.7986	44.5020	.022471
25	.29530	3.38635	14.0939	47.7271	.020952
26	.28124	3.55567	14.3752	51.1135	.019564
27	.26785	3.73346	14.6430	54.6691	.018292
28	.25509	3.92013	14.8981	58.4026	.017123
29	.24295	4.11614	15.1411	62.3227	.016046
30	.23138	4.32194	15.3725	66.4388	.015051
31	.22036	4.53804	15.5928	70.7608	.014132
32	.20987	4.76494	15.8027	75.2988	.013280
33	.19987	5.00319	16.0025	80.0638	.012490
34	.19035	5.25335	16.1929	85.0670	.011755
35	.18129	5.51602	16.3742	90.3203	.011072
36	.17266	5.79182	16.5469	95.8363	.010434
37	.16444	6.08141	16.7113	101.6281	.009840
38	.15661	6.38548	16.8679	107.7095	.009284
39	.14915	6.70475	17.0170	114.0950	.008765
40	.14205	7.03999	17.1591	120.7998	.008278
41	.13528	7.39199	17.2944	127.8398	.007822
42	.12884	7.76159	17.4232	135.2318	.007395
43	.12270	8.14967	17.5459	142.9933	.006993
44	.11686	8.55715	17.6628	151.1430	.006616
45	.11130	8.98501	17.7741	159.7002	.006262
46	.10600	9.43426	17.8801	168.6852	.005928
47	.10095	9.90597	17.9810	178.1194	.005614
48	.09614	10.40127	18.0772	188.0254	.005318
49	.09156	10.92133	18.1687	198.4267	.005040
50	.08720	11.46740	18.2559	209.3480	.004777

Constants	
Function	Value
i	.060000
$i^{(2)}$	.059126
$i^{(4)}$	.058695
$i^{(12)}$	.058411
δ	.058269
d	.056604
$d^{(2)}$	.057428
$d^{(4)}$	.057847
$d^{(12)}$	.058128
δ	.058269
v	.943396
$v^{1/2}$	.971286
$v^{1/4}$	.985538
$v^{1/12}$	.995156
$1+i$	1.060000
$(1+i)^{1/2}$	1.029563
$(1+i)^{1/4}$	1.014674
$(1+i)^{1/12}$	1.004868
$i/i^{(2)}$	1.014782
$i/i^{(4)}$	1.022227
$i/i^{(12)}$	1.027211
i/δ	1.029709
$i/d^{(2)}$	1.044782
$i/d^{(4)}$	1.037227
$i/d^{(12)}$	1.032211
i/δ	1.029709

INTEREST TABLES AT 6%

n	v^n	$(1+i)^n$	$a_{\overline{n}\mid}$	$s_{\overline{n}\mid}$	$1/s_{\overline{n}\mid}$
1	.94340	1.06000	.9434	1.0000	1.000000
2	.89000	1.12360	1.8334	2.0600	.485437
3	.83962	1.19102	2.6730	3.1836	.314110
4	.79209	1.26248	3.4651	4.3746	.228591
5	.74726	1.33823	4.2124	5.6371	.177396
6	.70496	1.41852	4.9173	6.9753	.143363
7	.66506	1.50363	5.5824	8.3938	.119135
8	.62741	1.59385	6.2098	9.8975	.101036
9	.59190	1.68948	6.8017	11.4913	.087022
10	.55839	1.79085	7.3601	13.1808	.075868
11	.52679	1.89830	7.8869	14.9716	.066793
12	.49697	2.01220	8.3838	16.8699	.059277
13	.46884	2.13293	8.8527	18.8821	.052960
14	.44230	2.26090	9.2950	21.0151	.047585
15	.41727	2.39656	9.7122	23.2760	.042963
16	.39365	2.54035	10.1059	25.6725	.038952
17	.37136	2.69277	10.4773	28.2129	.035445
18	.35034	2.85434	10.8276	30.9057	.032357
19	.33051	3.02560	11.1581	33.7600	.029621
20	.31180	3.20714	11.4699	36.7856	.027185
21	.29416	3.39956	11.7641	39.9927	.025005
22	.27751	3.60354	12.0416	43.3923	.023046
23	.26180	3.81975	12.3034	46.9958	.021278
24	.24698	4.04893	12.5504	50.8156	.019679
25	.23300	4.29187	12.7834	54.8645	.018227
26	.21981	4.54938	13.0032	59.1564	.016904
27	.20737	4.82235	13.2105	63.7058	.015697
28	.19563	5.11169	13.4062	68.5281	.014593
29	.18456	5.41839	13.5907	73.6398	.013580
30	.17411	5.74349	13.7648	79.0582	.012649
31	.16425	6.08810	13.9291	84.8017	.011792
32	.15496	6.45339	14.0840	90.8898	.011002
33	.14619	6.84059	14.2302	97.3432	.010273
34	.13791	7.25103	14.3681	104.1838	.009598
35	.13011	7.68609	14.4982	111.4348	.008974
36	.12274	8.14725	14.6210	119.1209	.008395
37	.11579	8.63609	14.7368	127.2681	.007857
38	.10924	9.15425	14.8460	135.9042	.007358
39	.10306	9.70351	14.9491	145.0585	.006894
40	.09722	10.28572	15.0463	154.7620	.006462
41	.09172	10.90286	15.1380	165.0477	.006059
42	.08653	11.55703	15.2245	175.9505	.005683
43	.08163	12.25045	15.3062	187.5076	.005333
44	.07701	12.98548	15.3832	199.7580	.005006
45	.07265	13.76461	15.4558	212.7435	.004700
46	.06854	14.59049	15.5244	226.5081	.004415
47	.06466	15.46592	15.5890	241.0986	.004148
48	.06100	16.39387	15.6500	256.5645	.003898
49	.05755	17.37750	15.7076	272.9584	.003664
50	.05429	18.42015	15.7619	290.3359	.003444

INTEREST TABLES AT 7%

Constants	
Function	Value
i	.070000
$i^{(2)}$	.068816
$i^{(4)}$	.068234
$i^{(12)}$	.067850
δ	.067659
d	.065421
$d^{(2)}$	.066527
$d^{(4)}$	.067090
$d^{(12)}$	.067468
δ	.067659
v	.934579
$v^{1/2}$	.966736
$v^{1/4}$	.983228
$v^{1/12}$	.994378
$1+i$	1.070000
$(1+i)^{1/2}$	1.034408
$(1+i)^{1/4}$	1.017059
$(1+i)^{1/12}$	1.005654
$i/i^{(2)}$	1.017204
$i/i^{(4)}$	1.025880
$i/i^{(12)}$	1.031691
i/δ	1.034605
$i/d^{(2)}$	1.052204
$i/d^{(4)}$	1.043380
$i/d^{(12)}$	1.037525
i/δ	1.034605

n	v^n	$(1+i)^n$	$a_{\overline{n}\rvert}$	$s_{\overline{n}\rvert}$	$1/s_{\overline{n}\rvert}$
1	.93458	1.07000	.9346	1.0000	1.000000
2	.87344	1.14490	1.8080	2.0700	.483092
3	.81630	1.22504	2.6243	3.2149	.311052
4	.76290	1.31080	3.3872	4.4399	.225228
5	.71299	1.40255	4.1002	5.7507	.173891
6	.66634	1.50073	4.7665	7.1533	.139796
7	.62275	1.60578	5.3893	8.6540	.115553
8	.58201	1.71819	5.9713	10.2598	.097468
9	.54393	1.83846	6.5152	11.9780	.083486
10	.50835	1.96715	7.0236	13.8164	.072378
11	.47509	2.10485	7.4987	15.7836	.063357
12	.44401	2.25219	7.9427	17.8885	.055902
13	.41496	2.40985	8.3577	20.1406	.049651
14	.38782	2.57853	8.7455	22.5505	.044345
15	.36245	2.75903	9.1079	25.1290	.039795
16	.33873	2.95216	9.4466	27.8881	.035858
17	.31657	3.15882	9.7632	30.8402	.032425
18	.29586	3.37993	10.0591	33.9990	.029413
19	.27651	3.61653	10.3356	37.3790	.026753
20	.25842	3.86968	10.5940	40.9955	.024393
21	.24151	4.14056	10.8355	44.8652	.022289
22	.22571	4.43040	11.0612	49.0057	.020406
23	.21095	4.74053	11.2722	53.4361	.018714
24	.19715	5.07237	11.4693	58.1767	.017189
25	.18425	5.42743	11.6536	63.2490	.015811
26	.17220	5.80735	11.8258	68.6765	.014561
27	.16093	6.21387	11.9867	74.4838	.013426
28	.15040	6.64884	12.1371	80.6977	.012392
29	.14056	7.11426	12.2777	87.3465	.011449
30	.13137	7.61226	12.4090	94.4608	.010586
31	.12277	8.14511	12.5318	102.0730	.009797
32	.11474	8.71527	12.6466	110.2182	.009073
33	.10723	9.32534	12.7538	118.9334	.008408
34	.10022	9.97811	12.8540	128.2588	.007797
35	.09366	10.67658	12.9477	138.2369	.007234
36	.08754	11.42394	13.0352	148.9135	.006715
37	.08181	12.22362	13.1170	160.3374	.006237
38	.07646	13.07927	13.1935	172.5610	.005795
39	.07146	13.99482	13.2649	185.6403	.005387
40	.06678	14.97446	13.3317	199.6351	.005009
41	.06241	16.02267	13.3941	214.6096	.004660
42	.05833	17.14426	13.4524	230.6322	.004336
43	.05451	18.34435	13.5070	247.7765	.004036
44	.05095	19.62846	13.5579	266.1209	.003758
45	.04761	21.00245	13.6055	285.7493	.003500
46	.04450	22.47262	13.6500	306.7518	.003260
47	.04159	24.04571	13.6916	329.2244	.003037
48	.03887	25.72891	13.7305	353.2701	.002831
49	.03632	27.52993	13.7668	378.9990	.002639
50	.03395	29.45703	13.8007	406.5289	.002460

INTEREST TABLES AT 8%

Constants	
Function	Value
i	.080000
$i^{(2)}$	.078461
$i^{(4)}$	.077706
$i^{(12)}$	.077208
δ	.076961
d	.074074
$d^{(2)}$	.075499
$d^{(4)}$	.076225
$d^{(12)}$	.076715
δ	.076961
v	.925926
$v^{1/2}$	.962250
$v^{1/4}$	.980944
$v^{1/12}$	.993607
$1+i$	1.080000
$(1+i)^{1/2}$	1.039230
$(1+i)^{1/4}$	1.019427
$(1+i)^{1/12}$	1.006434
$i/i^{(2)}$	1.019615
$i/i^{(4)}$	1.029519
$i/i^{(12)}$	1.036157
i/δ	1.039487
$i/d^{(2)}$	1.059615
$i/d^{(4)}$	1.049519
$i/d^{(12)}$	1.042824
i/δ	1.039487

n	v^n	$(1+i)^n$	$a_{\overline{n}\rvert}$	$s_{\overline{n}\rvert}$	$1/s_{\overline{n}\rvert}$
1	.92593	1.08000	.9259	1.0000	1.000000
2	.85734	1.16640	1.7833	2.0800	.480769
3	.79383	1.25971	2.5771	3.2464	.308034
4	.73503	1.36049	3.3121	4.5061	.221921
5	.68058	1.46933	3.9927	5.8666	.170456
6	.63017	1.58687	4.6229	7.3359	.136315
7	.58349	1.71382	5.2064	8.9228	.112072
8	.54027	1.85093	5.7466	10.6366	.094015
9	.50025	1.99900	6.2469	12.4876	.080080
10	.46319	2.15892	6.7101	14.4866	.069029
11	.42888	2.33164	7.1390	16.6455	.060076
12	.39711	2.51817	7.5361	18.9771	.052695
13	.36770	2.71962	7.9038	21.4953	.046522
14	.34046	2.93719	8.2442	24.2149	.041297
15	.31524	3.17217	8.5595	27.1521	.036830
16	.29189	3.42594	8.8514	30.3243	.032977
17	.27027	3.70002	9.1216	33.7502	.029629
18	.25025	3.99602	9.3719	37.4502	.026702
19	.23171	4.31570	9.6036	41.4463	.024128
20	.21455	4.66096	9.8181	45.7620	.021852
21	.19866	5.03383	10.0168	50.4229	.019832
22	.18394	5.43654	10.2007	55.4568	.018032
23	.17032	5.87146	10.3711	60.8933	.016422
24	.15770	6.34118	10.5288	66.7648	.014978
25	.14602	6.84848	10.6748	73.1059	.013679
26	.13520	7.39635	10.8100	79.9544	.012507
27	.12519	7.98806	10.9352	87.3508	.011448
28	.11591	8.62711	11.0511	95.3388	.010489
29	.10733	9.31727	11.1584	103.9659	.009619
30	.09938	10.06266	11.2578	113.2832	.008827
31	.09202	10.86767	11.3498	123.3459	.008107
32	.08520	11.73708	11.4350	134.2135	.007451
33	.07889	12.67605	11.5139	145.9506	.006852
34	.07305	13.69013	11.5869	158.6267	.006304
35	.06763	14.78534	11.6546	172.3168	.005803
36	.06262	15.96817	11.7172	187.1021	.005345
37	.05799	17.24563	11.7752	203.0703	.004924
38	.05369	18.62528	11.8289	220.3159	.004539
39	.04971	20.11530	11.8786	238.9412	.004185
40	.04603	21.72452	11.9246	259.0565	.003860
41	.04262	23.46248	11.9672	280.7810	.003561
42	.03946	25.33948	12.0067	304.2435	.003287
43	.03654	27.36664	12.0432	329.5830	.003034
44	.03383	29.55597	12.0771	356.9496	.002802
45	.03133	31.92045	12.1084	386.5056	.002587
46	.02901	34.47409	12.1374	418.4261	.002390
47	.02686	37.23201	12.1643	452.9002	.002208
48	.02487	40.21057	12.1891	490.1322	.002040
49	.02303	43.42742	12.2122	530.3427	.001886
50	.02132	46.90161	12.2335	573.7702	.001743

INTEREST TABLES AT 9%

Constants	
Function	**Value**
i	.090000
$i^{(2)}$	.088061
$i^{(4)}$	.087113
$i^{(12)}$	.086488
δ	.086178
d	.082569
$d^{(2)}$	.084347
$d^{(4)}$	.085256
$d^{(12)}$	.085869
δ	.086178
v	.917431
$v^{1/2}$	.957826
$v^{1/4}$	.978686
$v^{1/12}$	.992844
$1+i$	1.090000
$(1+i)^{1/2}$	1.044031
$(1+i)^{1/4}$	1.021778
$(1+i)^{1/12}$	1.007207
$i/i^{(2)}$	1.022015
$i/i^{(4)}$	1.033144
$i/i^{(12)}$	1.040608
i/δ	1.044354
$i/d^{(2)}$	1.067015
$i/d^{(4)}$	1.055644
$i/d^{(12)}$	1.048108
i/δ	1.044354

n	v^n	$(1+i)^n$	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$1/s_{\overline{n}\vert}$
1	.91743	1.09000	.9174	1.0000	1.000000
2	.84168	1.18810	1.7591	2.0900	.478469
3	.77218	1.29503	2.5313	3.2781	.305055
4	.70843	1.41158	3.2397	4.5731	.218669
5	.64993	1.53862	3.8897	5.9847	.167092
6	.59627	1.67710	4.4859	7.5233	.132920
7	.54703	1.82804	5.0330	9.2004	.108691
8	.50187	1.99256	5.5348	11.0285	.090674
9	.46043	2.17189	5.9952	13.0210	.076799
10	.42241	2.36736	6.4177	15.1929	.065820
11	.38753	2.58043	6.8052	17.5603	.056947
12	.35553	2.81266	7.1607	20.1407	.049651
13	.32618	3.06580	7.4869	22.9534	.043567
14	.29925	3.34173	7.7862	26.0192	.038433
15	.27454	3.64248	8.0607	29.3609	.034059
16	.25187	3.97031	8.3126	33.0034	.030300
17	.23107	4.32763	8.5436	36.9737	.027046
18	.21199	4.71712	8.7556	41.3013	.024212
19	.19449	5.14166	8.9501	46.0185	.021730
20	.17843	5.60441	9.1285	51.1601	.019546
21	.16370	6.10881	9.2922	56.7645	.017617
22	.15018	6.65860	9.4424	62.8733	.015905
23	.13778	7.25787	9.5802	69.5319	.014382
24	.12640	7.91108	9.7066	76.7898	.013023
25	.11597	8.62308	9.8226	84.7009	.011806
26	.10639	9.39916	9.9290	93.3240	.010715
27	.09761	10.24508	10.0266	102.7231	.009735
28	.08955	11.16714	10.1161	112.9682	.008852
29	.08215	12.17218	10.1983	124.1354	.008056
30	.07537	13.26768	10.2737	136.3075	.007336
31	.06915	14.46177	10.3428	149.5752	.006686
32	.06344	15.76333	10.4062	164.0370	.006096
33	.05820	17.18203	10.4644	179.8003	.005562
34	.05339	18.72841	10.5178	196.9823	.005077
35	.04899	20.41397	10.5668	215.7108	.004636
36	.04494	22.25123	10.6118	236.1247	.004235
37	.04123	24.25384	10.6530	258.3759	.003870
38	.03783	26.43668	10.6908	282.6298	.003538
39	.03470	28.81598	10.7255	309.0665	.003236
40	.03184	31.40942	10.7574	337.8824	.002960
41	.02921	34.23627	10.7866	369.2919	.002708
42	.02680	37.31753	10.8134	403.5281	.002478
43	.02458	40.67611	10.8380	440.8457	.002268
44	.02255	44.33696	10.8605	481.5218	.002077
45	.02069	48.32729	10.8812	525.8587	.001902
46	.01898	52.67674	10.9002	574.1860	.001742
47	.01742	57.41765	10.9176	626.8628	.001595
48	.01598	62.58524	10.9336	684.2804	.001461
49	.01466	68.21791	10.9482	746.8656	.001339
50	.01345	74.35752	10.9617	815.0836	.001227

INTEREST TABLES AT 10%

Constants

Function	Value
i	.100000
$i^{(2)}$	.097618
$i^{(4)}$	.096455
$i^{(12)}$	.095690
δ	.095310
d	.090909
$d^{(2)}$	.093075
$d^{(4)}$	.094184
$d^{(12)}$	.094933
δ	.095310
v	.909091
$v^{1/2}$	.953463
$v^{1/4}$	.976454
$v^{1/12}$	.992089
$1+i$	1.100000
$(1+i)^{1/2}$	1.048809
$(1+i)^{1/4}$	1.024114
$(1+i)^{1/12}$	1.007974
$i/i^{(2)}$	1.024404
$i/i^{(4)}$	1.036756
$i/i^{(12)}$	1.045045
i/δ	1.049206
$i/d^{(2)}$	1.074404
$i/d^{(4)}$	1.061756
$i/d^{(12)}$	1.053378
i/δ	1.049206

n	v^n	$(1+i)^n$	$a_{\overline{n}\vert}$	$s_{\overline{n}\vert}$	$1/s_{\overline{n}\vert}$
1	.90909	1.10000	.9091	1.0000	1.000000
2	.82645	1.21000	1.7355	2.1000	.476190
3	.75131	1.33100	2.4869	3.3100	.302115
4	.68301	1.46410	3.1699	4.6410	.215471
5	.62092	1.61051	3.7908	6.1051	.163797
6	.56447	1.77156	4.3553	7.7156	.129607
7	.51316	1.94872	4.8684	9.4872	.105405
8	.46651	2.14359	5.3349	11.4359	.087444
9	.42410	2.35795	5.7590	13.5795	.073641
10	.38554	2.59374	6.1446	15.9374	.062745
11	.35049	2.85312	6.4951	18.5312	.053963
12	.31863	3.13843	6.8137	21.3843	.046763
13	.28966	3.45227	7.1034	24.5227	.040779
14	.26333	3.79750	7.3667	27.9750	.035746
15	.23939	4.17725	7.6061	31.7725	.031474
16	.21763	4.59497	7.8237	35.9497	.027817
17	.19784	5.05447	8.0216	40.5447	.024664
18	.17986	5.55992	8.2014	45.5992	.021930
19	.16351	6.11591	8.3649	51.1591	.019547
20	.14864	6.72750	8.5136	57.2750	.017460
21	.13513	7.40025	8.6487	64.0025	.015624
22	.12285	8.14027	8.7715	71.4027	.014005
23	.11168	8.95430	8.8832	79.5430	.012572
24	.10153	9.84973	8.9847	88.4973	.011300
25	.09230	10.83471	9.0770	98.3471	.010168
26	.08391	11.91818	9.1609	109.1818	.009159
27	.07628	13.10999	9.2372	121.0999	.008258
28	.06934	14.42099	9.3066	134.2099	.007451
29	.06304	15.86309	9.3696	148.6309	.006728
30	.05731	17.44940	9.4269	164.4940	.006079
31	.05210	19.19434	9.4790	181.9434	.005496
32	.04736	21.11378	9.5264	201.1378	.004972
33	.04306	23.22515	9.5694	222.2515	.004499
34	.03914	25.54767	9.6086	245.4767	.004074
35	.03558	28.10244	9.6442	271.0244	.003690
36	.03235	30.91268	9.6765	299.1268	.003343
37	.02941	34.00395	9.7059	330.0395	.003030
38	.02673	37.40434	9.7327	364.0434	.002747
39	.02430	41.14478	9.7570	401.4478	.002491
40	.02209	45.25926	9.7791	442.5926	.002259
41	.02009	49.78518	9.7991	487.8518	.002050
42	.01826	54.76370	9.8174	537.6370	.001860
43	.01660	60.24007	9.8340	592.4007	.001688
44	.01509	66.26408	9.8491	652.6408	.001532
45	.01372	72.89048	9.8628	718.9048	.001391
46	.01247	80.17953	9.8753	791.7953	.001263
47	.01134	88.19749	9.8866	871.9749	.001147
48	.01031	97.01723	9.8969	960.1723	.001041
49	.00937	106.71896	9.9063	1057.1896	.000946
50	.00852	117.39085	9.9148	1163.9085	.000859

INTEREST TABLES AT 12%

Constants	
Function	Value
i	.120000
$i^{(2)}$	.116601
$i^{(4)}$	.114949
$i^{(12)}$	.113866
δ	.113329
d	.107143
$d^{(2)}$	.110178
$d^{(4)}$	.111738
$d^{(12)}$	.112795
δ	.113329
v	.892857
$v^{1/2}$	.944911
$v^{1/4}$	.972065
$v^{1/12}$	.990600
$1+i$	1.120000
$(1+i)^{1/2}$	1.058301
$(1+i)^{1/4}$	1.028737
$(1+i)^{1/12}$	1.009489
$i/i^{(2)}$	1.029150
$i/i^{(4)}$	1.043938
$i/i^{(12)}$	1.053875
i/δ	1.058867
$i/d^{(2)}$	1.089150
$i/d^{(4)}$	1.073938
$i/d^{(12)}$	1.063875
i/δ	1.058867

n	v^n	$(1+i)^n$	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$	$1/s_{\overline{n}\rceil}$
1	.89286	1.12000	.8929	1.0000	1.000000
2	.79719	1.25440	1.6901	2.1200	.471698
3	.71178	1.40493	2.4018	3.3744	.296349
4	.63552	1.57352	3.0373	4.7793	.209234
5	.56743	1.76234	3.6048	6.3528	.157410
6	.50663	1.97382	4.1114	8.1152	.123226
7	.45235	2.21068	4.5638	10.0890	.099118
8	.40388	2.47596	4.9676	12.2997	.081303
9	.36061	2.77308	5.3282	14.7757	.067679
10	.32197	3.10585	5.6502	17.5487	.056984
11	.28748	3.47855	5.9377	20.6546	.048415
12	.25668	3.89598	6.1944	24.1331	.041437
13	.22917	4.36349	6.4235	28.0291	.035677
14	.20462	4.88711	6.6282	32.3926	.030871
15	.18270	5.47357	6.8109	37.2797	.026824
16	.16312	6.13039	6.9740	42.7533	.023390
17	.14564	6.86604	7.1196	48.8837	.020457
18	.13004	7.68997	7.2497	55.7497	.017937
19	.11611	8.61276	7.3658	63.4397	.015763
20	.10367	9.64629	7.4694	72.0524	.013879
21	.09256	10.80385	7.5620	81.6987	.012240
22	.08264	12.10031	7.6446	92.5026	.010811
23	.07379	13.55235	7.7184	104.6029	.009560
24	.06588	15.17863	7.7843	118.1552	.008463
25	.05882	17.00006	7.8431	133.3339	.007500
26	.05252	19.04007	7.8957	150.3339	.006652
27	.04689	21.32488	7.9426	169.3740	.005904
28	.04187	23.88387	7.9844	190.6989	.005244
29	.03738	26.74993	8.0218	214.5828	.004660
30	.03338	29.95992	8.0552	241.3327	.004144
31	.02980	33.55511	8.0850	271.2926	.003686
32	.02661	37.58173	8.1116	304.8477	.003280
33	.02376	42.09153	8.1354	342.4294	.002920
34	.02121	47.14252	8.1566	384.5210	.002601
35	.01894	52.79962	8.1755	431.6635	.002317
36	.01691	59.13557	8.1924	484.4631	.002064
37	.01510	66.23184	8.2075	543.5987	.001840
38	.01348	74.17966	8.2210	609.8305	.001640
39	.01204	83.08122	8.2330	684.0102	.001462
40	.01075	93.05097	8.2438	767.0914	.001304
41	.00960	104.21709	8.2534	860.1424	.001163
42	.00857	116.72314	8.2619	964.3595	.001037
43	.00765	130.72991	8.2696	1081.0826	.000925
44	.00683	146.41750	8.2764	1211.8125	.000825
45	.00610	163.98760	8.2825	1358.2300	.000736
46	.00544	183.66612	8.2880	1522.2176	.000657
47	.00486	205.70605	8.2928	1705.8838	.000586
48	.00434	230.39078	8.2972	1911.5898	.000523
49	.00388	258.03767	8.3010	2141.9806	.000467
50	.00346	289.00219	8.3045	2400.0182	.000417

The Basics of Annuity Theory

A series of payments made at equal intervals of time is called an **annuity.** Common examples are house rents, mortgage payments on homes, and installments payments on automobiles.
An annuity where payments are guaranteed to occur for a fixed period of time is called an **annuity−certain.** For example, mortgage payments on a home. The fixed period of time for which payments are made is called **term** of the annuity. For example, in the case of a home mortgage a term can be either a 15−year loan or a 30−year loan.
Annuities that are not certain are called **contingent annuities.** For example, a pension is paid so long as the person survives. That is, regular payments are made as long as the person is alive. Pension is an example of contingent annuity also called **life annuity.**
Unless otherwise indicated, the annuity−certain is the type of annuity we will assume in this book, and the "certain" will be dropped from the name.
The interval between annuity payments is called a **payment period,** often just called a **period.** In Sections 15 through 21, we consider annuities for which the payment period and the interest conversion period are equal. Also, payments are of level amount, i.e. have the same fixed monetary value for each period. In Sections 22−29, we will discuss annuities for which payments are made more or less frequently than interest is converted and annuities with varying payments.
With level annuities, we will most of the time assume payments of 1 since any other level annuity can be obtained from this by a simple multiplication.

15 Present and Accumulated Values of an Annuity-Immediate

An annuity under which payments of 1 are made at the end of each period for n periods is called an **annuity−immediate** or **ordinary annuity.** The cash stream represented by the annuity can be visualized on a time diagram as shown in Figure 15.1 .

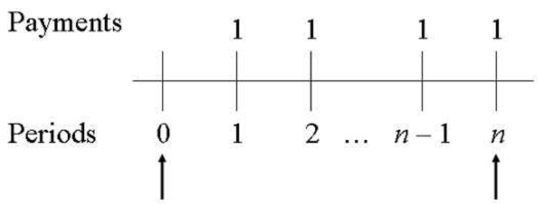


Figure 15.1

The first arrow shows the beginning of the first period, at the end of which the first payment is due under the annuity. The second arrow indicates the last payment date−just after the payment has been made.
Let i denote the interest rate per period. The present value of the annuity at time 0 will be denoted by $a_{\overline{n}|i}$ or simply $a_{\overline{n}|}$. See Figure 15.2.

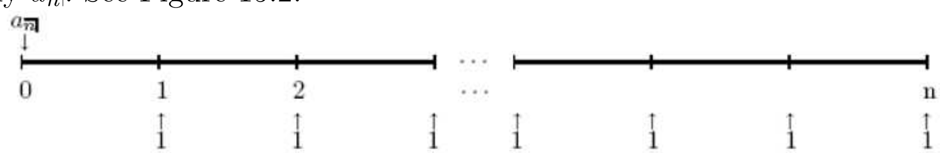


Figure 15.2

Using the equation of value with comparison date at time $t = 0$, we can write

$$a_{\overline{n}|} = \nu + \nu^2 + \cdots + \nu^n.$$

That is, the present value of the annutiy is the sum of the present values of each of the n payments. We recognize the expression on the right−hand side as a geometric progression. Thus, multiplying both sides by ν to obtain

$$\nu a_{\overline{n}|} = \nu^2 + \nu^3 + \cdots + \nu^n + \nu^{n+1}.$$

Subtracting this from the previous equation we find

$$(1 - \nu)a_{\overline{n}|} = \nu(1 - \nu^n).$$

Hence,

$$a_{\overline{n}|} = \nu \cdot \frac{1 - \nu^n}{1 - \nu} = \nu \cdot \frac{1 - \nu^n}{i\nu} = \frac{1 - (1 + i)^{-n}}{i}. \tag{15.1}$$

Example 15.1
Calculate the present value of an annuity−immediate of amount \$100 paid annually for 5 years at the rate of interest of 9%.

Solution.
The answer is $100a_{\overline{5}|} = 100\frac{1-(1.09)^{-5}}{0.09} \approx 388.97$ ■

Formula (15.1) can be rewritten as

$$1 = \nu^n + ia_{\overline{n}|}.$$

This last equation is the equation of value at time $t = 0$ of an investment of \$1 for n periods during which an interest of i is received at the end of each period and is reinvested at the same rate i, and at the end of the n periods the original investment of \$1 is returned. Figure 15.3 describes the time diagram of this transaction.

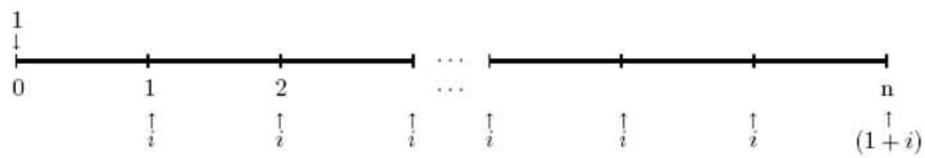


Figure 15.3

Next, we will determine the accumulated value of an annuity−immediate right after the nth payment is made. It is denoted by $s_{\overline{n}|}$. See Figure 15.4.

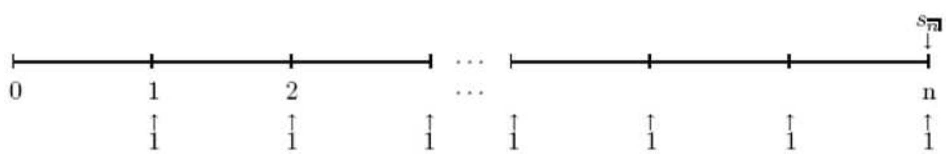


Figure 15.4

Writing the equation of value at the comparison date $t = n$ we find

$$s_{\overline{n}|} = 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1}.$$

That is, $s_{\overline{n}|}$ is the sum of the accumulated value of each of the n payments.
Using the definition of $s_{\overline{n}|}$ and the reasoning used to establish the formula for $a_{\overline{n}|}$ we can write

$$\begin{aligned} s_{\overline{n}|} =& 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} \\ =& \frac{(1+i)^n - 1}{(1+i) - 1} = \frac{(1+i)^n - 1}{i}. \end{aligned}$$

This last equation is equivalent to

$$1 + is_{\overline{n}|} = (1+i)^n.$$

This last equation is the equation of value at time $t = n$ of an investment of \$1 for n periods during which an interest of i is received at the end of each period and is reinvested at the same rate i, and at the end of the n periods the original investment of \$1 is returned. Figure 15.5 describes the time diagram of this transaction.

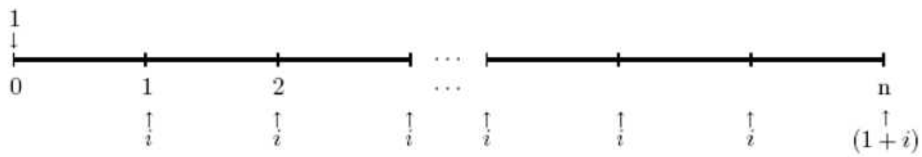


Figure 15.5

Example 15.2
Calculate the future value of an annuity−immediate of amount \$100 paid annually for 5 years at the rate of interest of 9%.

Solution.
The answer is $100s_{\overline{5}|} = 100 \times \frac{(1.09)^5-1}{0.09} \approx \$598.47$ ■

Example 15.3
Show that $a_{\overline{m+n}|} = a_{\overline{m}|} + \nu^m a_{\overline{n}|} = a_{\overline{n}|} + \nu^n a_{\overline{m}|}$. Interpret this result verbally.

Solution.
We have $a_{\overline{m}|} + \nu^m a_{\overline{n}|} = \frac{1-\nu^m}{i} + \nu^m \cdot \frac{1-\nu^n}{i} = \frac{1-\nu^m+\nu^m-\nu^{m+n}}{i} = \frac{1-\nu^{m+n}}{i} = a_{\overline{m+n}|}$.
A verbal interpretation is as follows: The present value of the first m payments of an $(m+n)-$year annuity− immediate of 1 is $a_{\overline{m}|}$. The remaining n payments have value $a_{\overline{n}|}$ at time $t = m$; discounted to the present, these are worth $\nu^m a_{\overline{n}|}$ at time $t = 0$ ■

Example 15.4
At an effective annual interest rate i, you are given
(1) the present value of an annuity immediate with annual payments of 1 for n years is 40
(2) the present value of an annuity immediate with annual payments of 1 for $3n$ years is 70.
Calculate the accumulated value of an annuity immediate with annual payments of 1 for $2n$ years.

Solution.
Using Example 15.3 we can write $a_{\overline{3n}|} = a_{\overline{2n}|} + \nu^{2n} a_{\overline{n}|} = a_{\overline{n}|} + \nu^n a_{\overline{n}|} + \nu^{2n} a_{\overline{n}|} = a_{\overline{n}|}(1 + \nu^n + \nu^{2n})$. Hence, we obtain the quadratic equation $\nu^{2n} + \nu^n + 1 = \frac{7}{4}$. Solving this equation we find $\nu^n = \frac{1}{2}$.

Again, using Example 15.3 we can write $a_{\overline{3n}|} = a_{\overline{n}|} + \nu^n a_{\overline{2n}|}$ which implies that $\nu^n a_{\overline{2n}|} = 70 - 40 = 30$ and therefore $a_{\overline{2n}|} = 60$. Finally, $s_{\overline{2n}|} = \nu^{-2n} a_{\overline{2n}|} = 4(60) = 240$ ■

We next establish a couple of relationships between $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

Theorem 15.1
With $a_{\overline{n}|}$ and $s_{\overline{n}|}$ as defined above we have
(i) $s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$. That is, the accumulated value of a principal of $a_{\overline{n}|}$ after n periods is just $s_{\overline{n}|}$.
(ii) $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$.

Proof.
(i) We have $s_{\overline{n}|} = \frac{(1+i)^n-1}{i} = (1+i)^n \cdot \frac{1-(1+i)^{-n}}{i} = (1+i)^n a_{\overline{n}|}$.
(ii) We have

$$\frac{1}{s_{\overline{n}|}} + i = \frac{i}{(1+i)^n - 1} + i = \frac{i + i(1+i)^n - i}{(1+i)^n - 1} = \frac{i}{1 - \nu^n} = \frac{1}{a_{\overline{n}|}} \ ■$$

A verbal interpretation of (ii) will be introduced when we discuss the concepts of amortization and sinking funds.

Example 15.5
For a given interest rate i, $a_{\overline{n}|} = 8.3064$ and $s_{\overline{n}|} = 14.2068$.
(a) Calculate i.
(b) Calculate n.

Solution.
(a) Using part (ii) of the previous theorem we find $i = \frac{1}{8.3064} - \frac{1}{14.2068} = 5\%$.
(b) Using part (i) of the previous theorem we find

$$n = \frac{1}{\ln(1+i)} \ln\left(\frac{s_{\overline{n}|}}{a_{\overline{n}|}}\right) = 11 \ ■$$

The type of annuity discussed in this section involves compound interest rate. It is possible to define annuity−immediate not involving compound interest such as simple interest rate, simple discount rate, and force of interest. For example, we wish to find the present value of an $n-$period annuity immediate in which each payment is invested at simple interest rate i. The present value is equal to the sum of the present value of the individual payments. Thus, we obtain

$$a_{\overline{n}|} = \frac{1}{1+i} + \frac{1}{1+2i} + \cdots + \frac{1}{1+ni}.$$

The accumulated value of such an annuity is equal to the accumulated value of the individual payments. That is,

$$s_{\overline{n}|} = 1 + (1+i) + (1+2i) + \cdots + [1+(n-1)i].$$

Example 15.6
Find an expression for $a_{\overline{n}|}$ assuming each payment of 1 is valued at simple discount rate d.

Solution.
The present value is the sum of the present value of individual payments. That is,

$$a_{\overline{n}|} = (1-d) + (1-2d) + \cdots + (1-nd) = n - d(1+2+\cdots+n) = n - \frac{n(n+1)}{2}d \blacksquare$$

Practice Problems

Problem 15.1
Consider an investment of \$5,000 at 6% convertible semiannually. How much can be withdrawn each half−year to use up the fund exactly at the end of 20 years?

Problem 15.2
The annual payment on a house is \$18,000. If payments are made for 40 years, how much is the house worth assuming annual interest rate of 6%?

Problem 15.3
If $d = 0.05$, calculate $a_{\overline{12}|}$.

Problem 15.4
Calculate the present value of 300 paid at the end of each year for 20 years using an annual effective interest rate of 8%.

Problem 15.5
If $a_{\overline{n}|} = x$ and $a_{\overline{2n}|} = y$, express d as a function of x and y.

Problem 15.6
(a) Given: $a_{\overline{7}|} = 5.153$, $a_{\overline{11}|} = 7.036$, $a_{\overline{18}|} = 9.180$. Find i.
(b) You are given that $a_{\overline{n}|} = 10.00$ and $a_{\overline{3n}|} = 24.40$. Determine $a_{\overline{4n}|}$.

Problem 15.7
Show that $s_{\overline{m+n}|} = s_{\overline{m}|} + (1+i)^m s_{\overline{n}|} = s_{\overline{n}|} + (1+i)^n s_{\overline{m}|}$. Interpret this result verbally.

Problem 15.8
A grandmother has a granddaughter entering university next year. Her granddaughter expects to remain in school for ten years and receive a PhD. This grandmother wishes to provide \$1,000 a year to her granddaughter for entertainment expenses. Assuming a 3.5% effective annual interest rate, how much does the grandmother have to deposit today to provide ten annual payments starting one year from now and continuing for ten years?

Problem 15.9
In the previous problem, if the PhD student saves the income from her grandmother in an account also paying 3.5% effective annual interest, how much will she have when she receives the final \$1,000 payment?

Problem 15.10
Find the present value of an annuity which pays $200 at the end of each quarter−year for 12 years if the rate of interest is 6% convertible quarterly.

Problem 15.11
Compare the total amount of interest that would be paid on a $3,000 loan over a 6−year period with an effective rate of interest of 7.5% per annum, under each of the following repayment plans:
(a) The entire loan plus accumulated interest is paid in one lump sum at the end of 6 years.
(b) Interest is paid each year as accrued, and the principal is repaid at the end of 6 years.
(c) The loan is repaid with level payments at the end of each year over the 6-year period.

Problem 15.12
A loan of $20,000 to purchase a car at annual rate of interest of 6% will be paid back through monthly installments over 5 years, with 1st installment to be made 1 month after the release of the loan. What is the monthly installment?

Problem 15.13
Over the next 20 years, you deposit money into a retirement account at the end of each year according to the following schedule:

Time	Amount invested each year
1 - 5	$2000
6 - 10	$3000
11 - 20	$5000

The effective annual rate of interest is 9%. Find the accumulated value of your account at time 20.

Problem 15.14
A family wishes to accumulate 50,000 in a college education fund by the end of 20 years. If they deposit 1,000 into the fund at the end of each of the first 10 years, and $1,000 + X$ at the end of each of the second 10 years, find X to the nearest unit if the fund earns 7% effective.

Problem 15.15
An annuity provides a payment of n at the end of each year for n years. The annual effective interest rate is $\frac{1}{n}$. What is the present value of the annuity?

Problem 15.16
The cash price of a new automobile is 10,000. The purchaser is willing to finance the car at 18% convertible monthly and to make payments of 250 at the end of each month for 4 years. Find the down payment which will be necessary.

Problem 15.17
You have $10,000 down payment on a $20,000 car. The dealer offers you the following two options:
(a) paying the balance with end−of−month payments over the next three years at $i^{(12)} = 0.12$;
(b) a reduction of $500 in the price of the car, the same down payment of $10,000, and bank financing of the balance after down payment, over 3 years with end−of−month payments at $i^{(12)} = 0.18$.
Which option is better?

Problem 15.18
Nancy has 10,000 in a bank account earning 6% compounded monthly. Calculate the amount that she can withdraw at the end of each month from the account if she wants to have zero in the account after 12 months.

Problem 15.19
Megan purchased a new car for 18,000. She finances the entire purchase over 60 months at a nominal rate of 12% compounded monthly. Calculate Megan's monthly payment.

Problem 15.20 ‡
Seth, Janice, and Lori each borrow 5,000 for five years at a nominal interest rate of 12%, compounded semiannually. Seth has interest accumulated over the five years and pays all the interest and principal in a lump sum at the end of five years. Janice pays interest at the end of every six−month period as it accrues and the principal at the end of five years. Lori repays her loan with 10 level payments at the end of every six−month period.
Calculate the total amount of interest paid on all three loans.

Problem 15.21
If $d^{(12)} = 12\%$, calculate the accumulated value of 100 paid at the end of each month for 12 months.

Problem 15.22
The accumulated value of an n year annuity−immediate is four times the present value of the same annuity. Calculate the accumulated value of 100 in $2n$ years.

Problem 15.23
You are given the following information:
(i) The present value of a $6n$−year annuity−immediate of 1 at the end of every year is 9.7578.
(ii) The present value of a $6n$−year annuity−immediate of 1 at the end of every second year is 4.760.
(iii) The present value of a $6n$−year annuity−immediate of 1 at the end of every third year is K.
Determine K assuming an annual effective interest rate of i.

Problem 15.24 ‡
Which of the following does not represent a definition of $a_{\overline{n}|}$?
(a) $\nu^n\left[\frac{(1+i)^n-1}{i}\right]$
(b) $\frac{1-\nu^n}{i}$
(c) $\nu+\nu^2+\cdots+\nu^n$
(d) $\nu\left[\frac{1-\nu^n}{1-\nu}\right]$
(e) $\frac{s_{\overline{n}|}}{(1+i)^{n-1}}$

Problem 15.25 ‡
To accumulate 8000 at the end of $3n$ years, deposits of 98 are made at the end of each of the first n years and 196 at the end of each of the next $2n$ years. The annual effective rate of interest is i. You are given $(1+i)^n = 2.0$. Determine i.

Problem 15.26 ‡
For 10,000, Kelly purchases an annuity−immediate that pays 400 quarterly for the next 10 years. Calculate the annual nominal interest rate convertible monthly earned by Kelly's investment. Hint: Use linear interpolation.

Problem 15.27 ‡
Susan and Jeff each make deposits of 100 at the end of each year for 40 years. Starting at the end of the 41st year, Susan makes annual withdrawals of X for 15 years and Jeff makes annual withdrawals of Y for 15 years. Both funds have a balance of 0 after the last withdrawal. Susan's fund earns an annual effective interest rate of 8%. Jeff's fund earns an annual effective interest rate of 10%. Calculate $Y - X$.

Problem 15.28
A loan of 10,000 is being repaid by 10 semiannual payments, with the first payment made one−half year after the loan. The first 5 payments are K each, and the final 5 are $K + 200$ each. What is K if $i^{(2)} = 0.06$?

Problem 15.29
Smith makes deposits of 1,000 on the last day of each month in an account earning interest at rate $i^{(12)} = 0.12$. The first deposit is January 31, 2005 and the final deposit is December 31, 2029. The accumulated account is used to make monthly payments of Y starting January 31, 2030 with the final one on December 31, 2054. Find Y.

Problem 15.30
A loan of \$1,000 is to be repaid with annual payments at the end of each year for the next 20 years. For the first five years the payments are k per year; the second 5 years, $2k$ per year; the third 5 years, $3k$ per year; and the fourth 5 years, $4k$ per year. Find an expression for k.

Problem 15.31 ‡
Happy and financially astute parents decide at the birth of their daughter that they will need to provide 50,000 at each of their daughter's 18th, 19th, 20th and 21st birthdays to fund her college education. They plan to contribute X at each of their daughter's 1st through 17th birthdays to fund the four 50,000 withdrawals. If they anticipate earning a constant 5% annual effective rate on their contributions, which of the following equations of value can be used to determine X, assuming compound interest?

(A) $X(\nu + \nu^2 + \cdots + \nu^{17}) = 50,000(\nu + \nu^2 + \nu^3 + \nu^4)$
(B) $X[(1.05)^{16} + (1.05)^{15} + \cdots + (1.05)] = 50,000(1 + \nu + \nu^2 + \nu^3)$
(C) $X[(1.05)^{17} + (1.05)^{16} + \cdots + (1.05) + 1] = 50,000(1 + \nu + \nu^2 + \nu^3)$
(D) $X[(1.05)^{17} + (1.05)^{16} + \cdots + (1.05)] = 50,000(1 + \nu + \nu^2 + \nu^3)$
(E) $X(1 + \nu + \nu^2 + \cdots + \nu^{17}) = 50,000(\nu^{18} + \nu^{19} + \nu^{20} + \nu^{21} + \nu^{22})$

Problem 15.32
For time $t > 0$, the discount function is defined by

$$[a(t)]^{-1} = \frac{1}{1 + 0.01t}.$$

Consider a five-year annuity with payments of 1 at times $t = 1, 2, 3, 4, 5$. Consider the following: A calculates $a_{\overline{5}|}$ as the sum of the present value of the individual payments. However, B accumulates the payments according to the accumulation function

$$a(t) = 1 + 0.01t$$

and then multiplies the result by $\frac{1}{a(5)}$. By how much do the answers of A and B differ?

Remark 15.1
It can be shown that the above two processes do not produce the same answers in general for any pattern of interest other than compound interest. That's why, it is always recommended to avoid dealing with annuities not involving compound interest, if possible.

Problem 15.33
Simplify the sum $\sum_{n=15}^{40} s_{\overline{n}|}$.

Problem 15.34
A 20 year annuity pays 100 every other year beginning at the end of the second year, with additional payments of 300 each at the ends of years 3, 9, and 15. The effective annual interest rate is 4%. Calculate the present value of the annuity.

Problem 15.35
An annuity pays 1 at the end of each 4−year period for 40 years. Given $a_{\overline{8}|i} = k$, find the present value of the annuity.

Problem 15.36
Smith is negotiating a price for a new car. He is willing to pay $250 at the end of each month for 60 months using the 4.9% compounded monthly interest rate that he qualifies for. Smith estimates that tax, title, and license for the new car will increase the negotiated price by 10%, and he estimates that he will receive $500 trade−in value for his current car. Calculate the highest negotiated price that Smith is willing to pay for the car.

Problem 15.37
An account is credited interest using 6% simple interest rate from the date of each deposit into the account. Annual payments of 100 are deposited into this account. Calculate the accumulated value of the account immediately after the 20^{th} deposit.

Problem 15.38
A homeowner signs a 30 year mortgage that requires payments of $971.27 at the end of each month. The interest rate on the mortgage is 6% compounded monthly. If the purchase price of the house is $180,000 then what percentage down payment was required?

Problem 15.39
A 25−year−old worker begins saving for retirement, making level annual deposits at the end of each year. The savings are invested at an annual effective interest rate of 8%, and are of an amount that is projected to equal 1,000,000 when the worker is 65 (after the 40th deposit is made on that date).
After making five annual deposits, the worker becomes unemployed for a period of time and, as a result, skips the next three annual deposits. Assuming that the account earns an 8% annual effective rate in all 40 years, what amount will the worker need to deposit in each of the remaining 32 years in order to achieve the original goal of a 1,000,000 balance at age 65?

Problem 15.40
Paul lends 8000 to Peter. Peter agrees to pay it back in 10 annual installments at 7% with the first payment due in one year. After making 4 payments, Peter renegotiates to pay off the debt with 4 additional annual payments. The new payments are calculated so that Paul will get a 6.5% annual yield over the entire 8−year period. Determine how much money Peter saved by renegotiating.

Problem 15.41
Mario deposits 100 into a fund at the end of each 2 year period for 20 years. The fund pays interest at an annual effective rate of i. The total amount of interest earned by the fund during the 19th and 20th years is 250. Calculate the accumulated amount in Marios account at the end of year 20.

Problem 15.42
Show that $\frac{1}{1-\nu^{10}} = \frac{1}{s_{\overline{10}|}}\left(s_{\overline{10}|} + \frac{1}{i}\right).$

Problem 15.43
If $a_{\overline{4}|} = 3.2397$ and $s_{\overline{4}|} = 4.5731$ what is the value of $a_{\overline{8}|}$?

Problem 15.44
Smith borrows \$5,000 on January 1, 2007. He repays the loan with 20 annual payments, starting January 1, 2008. The payments in even-number year are $2X$ each; the payments in odd-number years are X each. If $d = 0.08$, find the total amount of all 20 payments.

Problem 15.45
(a) Show that $a_{\overline{m-n}|} = a_{\overline{m}|} - \nu^m s_{\overline{n}|}$ where $0 < n < m$.
(b) Show that $s_{\overline{m-n}|} = s_{\overline{m}|} - (1+i)^m a_{\overline{n}|}$ where $0 < n < m$.

Problem 15.46
Show that $s_{\overline{n}|} \geq n \geq a_{\overline{n}|}$.

Thus far, we have only considered annuities where the payments are made at the end of the period, but it is possible that the circumstances may be such that the annuity is to run for a given number of periods and a portion of a period. The purpose of the remaining problems is to define terms such as $a_{\overline{n+k}|}$ and $s_{\overline{n+k}|}$ where n is a positive integer and $0 < k < 1$.

Problem 15.47
Let k be a positive real number. Find an expression for the error involved in approximating $\frac{(1+i)^k-1}{i}$ by the number k. Hint: Use the Taylor series expansion of $(1+i)^k$ around $i = 0$.

Problem 15.48
(a) Find the interest accrued of a principal of 1 at $t = n$ to time $t = n + k$, assuming compound interest.
(b) Show that

$$i(\nu + \nu^2 + \nu^3 + \cdots + \nu^n) + [(1+i)^k - 1]\nu^{n+k} + \nu^{n+k} = 1.$$

(c) Let $a_{\overline{n+\frac{1}{m}}|}$ denote the present value of an annuity consisting of n payments of 1 at the end of each period and a final payment of $\frac{(1+i)^{\frac{1}{m}}-1}{i}$ at the end of the $\frac{1}{m}$th fraction of the $(n+1)$ period. Define $a_{\overline{n+\frac{1}{m}}|} = \frac{1-\nu^{n+\frac{1}{m}}}{i}$. Show that

$$a_{\overline{n+\frac{1}{m}}|} = a_{\overline{n}|} + \nu^{n+\frac{1}{m}}\left[\frac{(1+i)^{\frac{1}{m}} - 1}{i}\right].$$

Thus, $a_{\overline{n+k}|}$ is the sum of the present value of an $n-$period annuity$-$immediate of 1 per period, plus a final payment at time $n+k$ of $\frac{(1+i)^k-1}{i}$.

Problem 15.49
Show that

$$a_{\overline{n+k}|} \approx a_{\overline{n}|} + k\nu^{n+\frac{1}{m}}.$$

Hint: Use Problems 15.47 and 15.48.

Problem 15.50
Compute $a_{\overline{5.25}|}$ if $i = 5\%$ using the following definitions:
(a) The formula established in Problem 15.48.
(b) A payment of 0.25 at time 5.25.
(c) A payment of 0.25 at time 6.

Problem 15.51
(a) Define $s_{\overline{n+k}|} = \frac{(1+i)^{n+k}-1}{k}$. Show that $s_{\overline{n+k}|} = (1+i)^k s_{\overline{n}|} + \frac{(1+i)^k-1}{i}$. Thus, $s_{\overline{n+k}|}$ can be interprested as the accumulated value of an $n-$period annuity$-$immediate at time $t = n+k$ with an additional payment of $\frac{(1+i)^k-1}{i}$ at time $t = n+k$.
(b) Show that $s_{\overline{n+k}|} \approx (1+i)^k s_{\overline{n}|} + k$.

16 Annuity in Advance: Annuity Due

An **annuity−due** is an annuity for which the payments are made at the beginning of the payment periods. The cash stream represented by the annuity can be visualized on a time diagram as shown in Figure 16.1 .

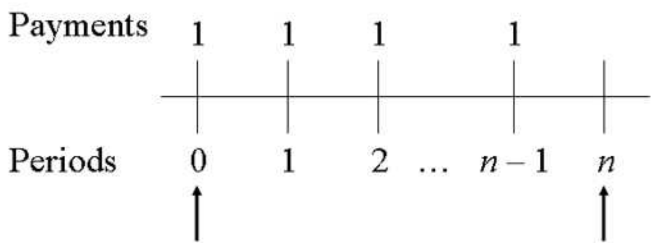


Figure 16.1

The first arrow shows the beginning of the first period at which the first payment is made under the annuity. The second arrow appears n periods after arrow 1, one period after the last payment is made.
Let i denote the interest rate per period. The present value of the annuity at time 0 will be denoted by $\ddot{a}_{\overline{n}|}$. To determine $\ddot{a}_{\overline{n}|}$, we can proceed as in the case of determining $a_{\overline{n}|}$. In this case, we use the time diagram shown in Figure 16.2

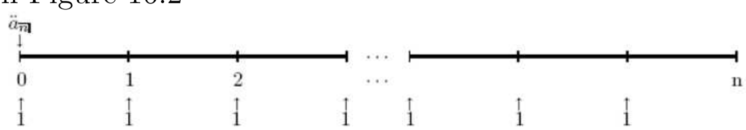


Figure 16.2

Considering the equation of value at time $t = 0$ we can write

$$\ddot{a}_{\overline{n}|} = 1 + \nu + \nu^2 + \cdots + \nu^{n-1}.$$

That $\ddot{a}_{\overline{n}|}$ is is equal to the sum of the present values of each of the n payments. We recognize the expression on the right−hand side as a geometric progression. Thus, multiplying both sides by ν to obtain

$$\nu\ddot{a}_{\overline{n}|} = \nu + \nu^2 + \nu^3 + \cdots + \nu^{n-1} + \nu^n.$$

Subtracting this from the previous equation we find

$$(1 - \nu)\ddot{a}_{\overline{n}|} = (1 - \nu^n).$$

Hence,

$$\ddot{a}_{\overline{n}|} = \frac{1-\nu^n}{1-\nu}.$$

Since $1-\nu = d$, we have

$$\ddot{a}_{\overline{n}|} = \frac{1-\nu^n}{d} = \frac{1-(1+i)^{-n}}{d} = \frac{1-(1-d)^n}{d}.$$

Example 16.1
Find $\ddot{a}_{\overline{8}|}$ if the effective rate of discount is 10%.

Solution.
Since $d = 0.10$, we have $\nu = 1-d = 0.9$. Hence, $\ddot{a}_{\overline{8}|} = \frac{1-(0.9)^8}{0.1} = 5.6953279$ ■

Example 16.2
What amount must you invest today at 6% interest rate compounded annually so that you can withdraw \$5,000 at the beginning of each year for the next 5 years?

Solution.
The answer is $5000\ddot{a}_{\overline{5}|} = 5000 \cdot \frac{1-(1.06)^{-5}}{0.06(1.06)^{-1}} = \$22,325.53$ ■

Now, let $\ddot{s}_{\overline{n}|}$ denote the accumulated value of an annuity−due at the end of n periods. To determine $\ddot{s}_{\overline{n}|}$ we proceed in a way analogous to determining $s_{\overline{n}|}$. We consider the time diagram shown in Figure 16.3.

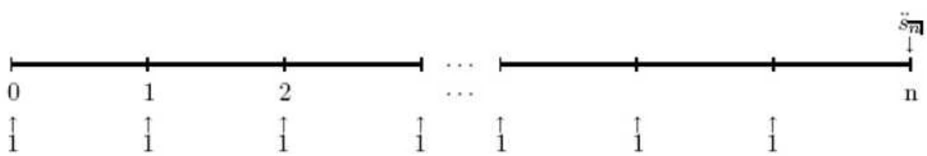


Figure 16.3

Writing the equation of value at time $t = n$ we find

$$\begin{aligned}\ddot{s}_{\overline{n}|} =&(1+i)+(1+i)^2+\cdots+(1+i)^{n-1}+(1+i)^n\\ =&(1+i)\frac{(1+i)^n-1}{(1+i)-1}\\ =&\frac{(1+i)^n-1}{i\nu} = \frac{(1+i)^n-1}{d}\end{aligned}$$

Example 16.3
What amount will accumulate if we deposit \$5,000 at the beginning of each year for the next 5 years? Assume an interest of 6% compounded annually.

Solution.
The answer is $5000\ddot{s}_{\overline{5}|} = 5000 \cdot \frac{(1.06)^5-1}{0.06(1.06)^{-1}} = \$29,876.59$ ■

The following theorem provides a couple of relationships between $\ddot{a}_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$

Theorem 16.1
(a) $\ddot{s}_{\overline{n}|} = (1+i)^n\ddot{a}_{\overline{n}|}$
(b) $\frac{1}{\ddot{a}_{\overline{n}|}} = \frac{1}{\ddot{s}_{\overline{n}|}} + d$

Proof.
(a) We have

$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d} = (1+i)^n \cdot \frac{1-(1+i)^{-n}}{d} = (1+i)^n\ddot{a}_{\overline{n}|}$$

(b)

$$\begin{aligned}\frac{1}{\ddot{s}_{\overline{n}|}} + d =& \frac{d}{(1+i)^n-1} + d\frac{(1+i)^n-1}{(1+i)^n-1}\\ =& \frac{d+d[(1+i)^n-1]}{(1+i)^n-1} = \frac{d(1+i)^n}{(1+i)^n-1}\\ =& \frac{d}{1-(1+i)^{-n}} = \frac{1}{\ddot{a}_{\overline{n}|}}\end{aligned}$$ ■

The result in (a) says that if the present value at time 0, $\ddot{a}_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $\ddot{s}_{\overline{n}|}$.

Example 16.4
For a given interest rate i, $\ddot{a}_{\overline{n}|} = 8.3064$ and $\ddot{s}_{\overline{n}|} = 14.2068$.
(a) Calculate d.
(b) Calculate n.

Solution.
(a) Using part (b) of the previous theorem we find $d = \frac{1}{8.3064} - \frac{1}{14.2068} = 5\%$.
(b) Using part (a) of the previous theorem we find

$$n = \frac{1}{\ln(1+i)}\ln\left(\frac{\ddot{s}_{\overline{n}|}}{\ddot{a}_{\overline{n}|}}\right) = 10.463$$ ■

It is possible to relate the annuity-immediate and the annuity-due as shown in the following theorem.

Theorem 16.2
(a) $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$.
(b) $\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}$.

Proof.
The two results can be obtained directly by using the time diagram shown in Figure 16.4

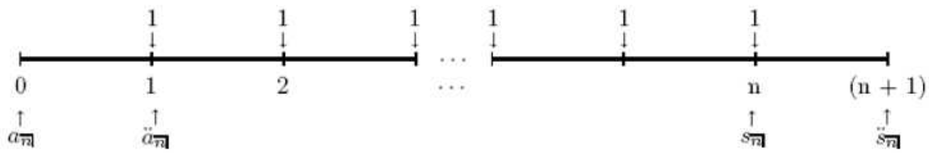


Figure 16.4

An algebraic justification is given next.
(a) Since $d = \frac{i}{i+1}$, we have $\ddot{a}_{\overline{n}|} = \frac{1-(1+i)^{-n}}{d} = (1+i)\cdot\frac{1-(1+i)^{-n}}{i} = (1+i)a_{\overline{n}|}$.
(b) Since $d = \frac{i}{i+1}$, we have: $\ddot{s}_{\overline{n}|} = \frac{(1+i)^n-1}{d} = (1+i)\cdot\frac{(1+i)^n-1}{i} = (1+i)s_{\overline{n}|}$ ■

An annuity−due starts one period earlier than an annuity−immediate and as a result, earns one more period of interest, hence more profit is made.

Example 16.5
Over the next 30 years, you deposit money into a retirement account at the beginning of each year. The first 10 payments are 200 each. The remaining 20 payments are 300 each. The effective annual rate of interest is 9%. Find the present value of these payments.

Solution.
The time diagram of this situation is shown in Figure 16.5

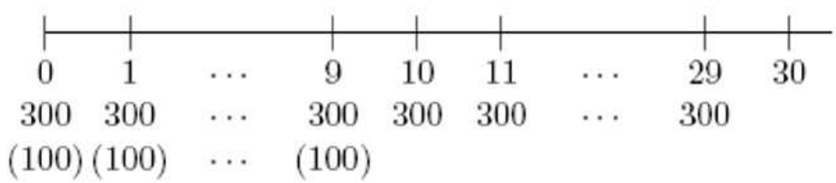


Figure 16.5

The answer is

$$300\ddot{a}_{\overline{30}|} - 100\ddot{a}_{\overline{10}|} = 1.09(300a_{\overline{30}|} - 100a_{\overline{10}|}) = \$2659.96 \blacksquare$$

More relationships between annuity-immediate and annuity-due are given below.

Theorem 16.3
(a) $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$.
(b) $s_{\overline{n}|} = \ddot{s}_{\overline{n-1}|} + 1$.

Proof.
(a) We have $\ddot{a}_{\overline{n}|} = \frac{1-(1+i)^{-n}}{d} = \frac{i+1}{i}[1-(1+i)^{-n}] = \frac{1-(1+i)^{-n+1}+i}{i} = 1 + a_{\overline{n-1}|}$.
The result has the following verbal interpretation: An additional payment of 1 at time 0 results in $a_{\overline{n-1}|}$ becoming n payments that now commence at the beginning of each year whose present value is $\ddot{a}_{\overline{n}|}$.
(b) We have $\ddot{s}_{\overline{n-1}|} = \frac{(1+i)^{n-1}-1}{d} = \frac{i+1}{i}[(1+i)^{n-1}-1] = \frac{(1+i)^n-1-i}{i} = s_{\overline{n}|} - 1$
An interpretation of (b) is as follows: A withdrawal of 1 at time n results in $s_{\overline{n}|}$ becoming $n-1$ payments that commence at the beginning of each year (starting at $t = 1$) whose accumulated value at time $t = n$ is $\ddot{s}_{\overline{n-1}|}$ ■

Remark 16.1
Most compound interest tables do not include values of annuities−due. Thus, the formulas of Theorems 16.2 - 16.3 must be used in finding numerical values for annuities−due.

Example 16.6
An investor wishes to accumulate $3000 at the end of 15 years in a fund which earns 8% effective. To accomplish this, the investor plans to make deposits at the beginning of each year, with the final payment to be made one year prior to the end of the investment period. How large should each deposit be?

Solution.
Let R be the payment at the beginning of each year. The accumulated value of the investment at the end of the investment period is to be $3000, so the equation of value at time $t = 15$ is $3000 = R\ddot{s}_{\overline{14}|}$. Solving for R we find

$$R = \frac{3000}{\ddot{s}_{\overline{14}|}} = \frac{3000}{s_{\overline{15}|} - 1} = \$114.71 ■$$

Example 16.7
Show that $\ddot{a}_{\overline{n}|} = a_{\overline{n}|} + 1 - \nu^n$. Interpret the result verbally.

Solution.
We have $\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|} = a_{\overline{n}|} + ia_{\overline{n}|} = a_{\overline{n}|} + i \cdot \frac{1-\nu^n}{i} = a_{\overline{n}|} + 1 - \nu^n$.
An additional payment of 1 at time 0 results in $a_{\overline{n}|}$ becoming $n+1$ payments that now commence at the beginning of the year and whose present value is $\ddot{a}_{\overline{n}|} + \nu^n$ ■

Example 16.8
Show that $\ddot{s}_{\overline{n}|} = s_{\overline{n}|} - 1 + (1+i)^n$. Interpret the result verbally.

Solution.
We have $\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|} = s_{\overline{n}|} + i \cdot \frac{(1+i)^n - 1}{i} = s_{\overline{n}|} - 1 + (1+i)^n$.
An additional payment of 1 at time n results in $\ddot{s}_{\overline{n}|}$ becoming $n+1$ payments that now commence at the beginning of the year and whose accumulated value at time $t = n$ is $s_{\overline{n}|} + (1+i)^n$ ■

Remark 16.2
The names annuity−immediate and annuity-due are used traditionally, although they do not seem to be logical. The first payment of an annuity−immediate is not made immediately at the beginning of the first payment period, it is due at the end of it.

Practice Problems

Problem 16.1
An 8-year annuity due has a present value of \$1,000. If the effective annual interest rate is 5%, then what is the value of the periodic payment?

Problem 16.2
An 8-year annuity due has a future value of \$1,000. Find the periodic payment of this annuity if the effective annual interest rate is 5%.

Problem 16.3
A 5-year annuity due has periodic cash flows of \$100 each year. Find the accumulated value of this annuity if the effective annual interest rate is 8%.

Problem 16.4
A 5-year annuity due has periodic cash flows of \$100 each year. Find the present value of this annuity if the effective annual interest rate is 8%.

Problem 16.5
Calculate the accumulated value immediately after the last payment of a 20 year annuity due of annual payments of 500 per year. The annual effective interest rate is 7%.

Problem 16.6
Megan wants to buy a car in 4 years for 18,000. She deposits X at the beginning of each month for four years into an account earning 6% compounded monthly. Calculate X.

Problem 16.7
Kathy wants to accumulate a sum of money at the end of 10 years to buy a house. In order to accomplish this goal, she can deposit 80 per month at the beginning of the month for the next ten years or 81 per month at the end of the month for the next ten years. Calculate the annual effective rate of interest earned by Kathy.

Problem 16.8
An annuity pays \$500 every six months for 5 years. The annual rate of interest is 8% convertible semiannually. Find each of the following:
(a) The PV of the annuity six months (one period) before the first payment,
(b) the PV of the annuity on the day of the first payment,
(c) the FV of the annuity on the day of the last payment,
(d) and the FV of the annuity six months after the last payment.

Problem 16.9
You will retire in 30 years. At the beginning of each month until you retire, you will invest X earning interest at 9% convertible monthly. Starting at year 30, you will withdraw \$4,000 at the beginning of each month for the next 15 years. Also, starting at year 30, your fund will only earn interest at 6% convertible monthly. Find X such that your account will be empty after the last withdrawal.

Problem 16.10 ‡
Chuck needs to purchase an item in 10 years. The item costs 200 today, but its price increases by 4% per year. To finance the purchase, Chuck deposits 20 into an account at the beginning of each year for 6 years. He deposits an additional X at the beginning of years 4, 5, and 6 to meet his goal. The annual effective interest rate is 10%. Calculate X.

Problem 16.11
Which of the following are true:
(i) $\ddot{s}_{\overline{9}|} + 1 = s_{\overline{10}|}$
(ii) $a_{\overline{10}|} - a_{\overline{3}|} = v^2 a_{\overline{7}|}$
(iii) $a_{\overline{5}|}(1+i) - 1 = a_{\overline{4}|}$?

Problem 16.12
Find the present value of payments of \$200 every six months starting immediately and continuing through four years from the present, and \$100 every six months thereafter through ten years from the present, if $i^{(2)} = 0.06$.

Problem 16.13
A worker aged 40 wishes to accumulate a fund for retirement by depositing \$1,000 at the beginning of each year for 25 years. Starting at age 65 the worker plans to make 15 annual withdrawals at the beginning of each year. Assuming all payments are certain to be made, find the amount of each withdrawal starting at age 65 to the nearest dollar, if the effective rate of interest is 8% during the first 25 years but only 7% thereafter.

Problem 16.14
Which one is greater $s_{\overline{n}|} - a_{\overline{n}|}$ or $\ddot{s}_{\overline{n}|} - \ddot{a}_{\overline{n}|}$?

Problem 16.15
A $(2n-1)-$payment annuity-immediate has payments $1, 2, \cdots, n-1, n, n-1, \cdots, 2, 1$. Show that the present value of the annuity one payment period before the first payment is $a_{\overline{n}|} \cdot \ddot{a}_{\overline{n}|}$.

Problem 16.16
Jeff makes payments at the end of each year into an account for 10 years. The present value of Jeff's payments is 5,000. Ryan makes a payment equal to Jeff's payment at the beginning of each year for 11 years into the same account. The present value of Ryan's payments is 5,900.
Calculate the amount of Jeff's payment.

Problem 16.17
A person deposits 100 at the beginning of each year for 20 years. Simple interest at an annual rate of i is credited to each deposit from the date of deposit to the end of the twenty year period. The total amount thus accumulated is 2,840. If instead, compound interest had been credited at an effective annual rate of i, what would the accumulated value of these deposits have been at the end of twenty years?

Problem 16.18
You plan to accumulate 100,000 at the end of 42 years by making the following deposits:
X at the beginning of years 1-14
No deposits at the beginning of years 15-32; and
Y at the beginning of years 33-42.
The annual effective interest rate is 7%.
Suppose $X - Y = 100$. Calculate Y.

Problem 16.19 ‡
Jim began saving money for his retirement by making monthly deposits of 200 into a fund earning 6% interest compounded monthly. The first deposit occurred on January 1, 1985. Jim became unemployed and missed making deposits 60 through 72. He then continued making monthly deposits of 200. How much did Jim accumulate in his fund on December 31, 1999 ?

Problem 16.20 ‡
An investor accumulates a fund by making payments at the beginning of each month for 6 years. Her monthly payment is 50 for the first 2 years, 100 for the next 2 years, and 150 for the last 2 years. At the end of the 7th year the fund is worth 10,000. The annual effective interest rate is i, and the monthly effective interest rate is j. Which of the following formulas represents the equation of value for this fund accumulation?
(a) $\ddot{s}_{\overline{24}|i}(1+i)[(1+i)^4+2(1+i)^2+3]=200$
(b) $\ddot{s}_{\overline{24}|i}(1+j)[(1+j)^4+2(1+j)^2+3]=200$
(c) $\ddot{s}_{\overline{24}|j}(1+i)[(1+i)^4+2(1+i)^2+3]=200$
(d) $s_{\overline{24}|j}(1+i)[(1+i)^4+2(1+i)^2+3]=200$
(e) $s_{\overline{24}|i}(1+j)[(1+j)^4+2(1+j)^2+3]=200$

Problem 16.21 ‡
Carol and John shared equally in an inheritance. Using his inheritance, John immediately bought a 10-year annuity-due with an annual payment of 2,500 each. Carol put her inheritance in an investment fund earning an annual effective interest rate of 9%. Two years later, Carol bought a 15-year annuity-immediate with annual payment of Z. The present value of both annuities was determined using an annual effective interest rate of 8%.
Calculate Z.

Problem 16.22 ‡
Jerry will make deposits of 450 at the end of each quarter for 10 years.
At the end of 15 years, Jerry will use the fund to make annual payments of Y at the beginning of each year for 4 years, after which the fund is exhausted.
The annual effective rate of interest is 7% . Determine Y.

Problem 16.23
If $a(t) = \frac{1}{\log_2(t+2) - \log_2(t+1)}$, find an expression for $\ddot{a}_{\overline{n}|}$ by directly taking the present value of the payments.

Problem 16.24
The accumulated value of an n year annuity-due is four times the present value of the same annuity. Calculate the accumulated value of 100 in $2n$ years.

Problem 16.25
Show that $s_{\overline{n}|} \cdot \ddot{a}_{\overline{n}|} > n^2$ for $i > 0$ and $n > 1$.

Problem 16.26 ‡
A man turns 40 today and wishes to provide supplemental retirement income of 3000 at the beginning of each month starting on his 65th birthday. Starting today, he makes monthly contributions of X to a fund for 25 years. The fund earns a nominal rate of 8% compounded monthly. Each 1000 will provide for 9.65 of income at the beginning of each month starting on his 65th birthday until the end of his life.
Calculate X.

Problem 16.27
An annuity pays 3 at the beginning of each 3 year period for 30 years. Find the accumulated value of the annuity just after the final payment, using $i^{(2)} = 0.06$.

Problem 16.28
A worker aged 30 wishes to accumulate a fund for retirement by depositing \$3,000 at the beginning

of each year for 35 years. Starting at age 65 the worker plans to make 20 equal annual withdrawals at the beginning of each year. Assuming that all payments are certain to be made, find the amount of each withdrawal starting at age 65, if the annual effective rate of interest is 9% during the first 35 years but only 6% thereafter.

Problem 16.29
Irene deposits 100 at the beginning of each year for 20 years into an account in which each deposit earns simple interest at a rate of 10% from the time of the deposit. Other than these deposits Irene makes no other deposits or withdrawals from the account until exactly 25 years after the first deposit was made, at which time she withdraws the full amount in the account.
Determine the amount of Irene's withdrawal.

Problem 16.30
Show that

$$\frac{\ddot{s}_{\overline{2n}|}}{\ddot{s}_{\overline{n}|}}+\frac{\ddot{s}_{\overline{n}|}}{\ddot{s}_{\overline{2n}|}}-\frac{\ddot{s}_{\overline{3n}|}}{\ddot{s}_{\overline{2n}|}}=1.$$

Problem 16.31
If $\ddot{a}_{\overline{p}|}=x$ and $s_{\overline{q}|}=y$, show that $a_{\overline{p+q}|}=\frac{\nu x+y}{1+iy}$.

17 Annuity Values on Any Date: Deferred Annuity

Evaluating annuities thus far has always been done at the beginning of the term(either on the date of, or one period before the first payment) or at the end of the term (either on the date of, or one period after the last payment). In this section, we shall now consider evaluating the
(1) present value of an annuity more than one period before the first payment date,
(2) accumulated value of an annuity more than one period after the last payment date,
(3) current value of an annuity between the first and last payment dates.
We will assume that the evaluation date is always an integral number of periods from each payment date.

(1) **Present values more than one period before the first payment date**
Consider the question of finding the present value of an annuity−immediate with periodic interest rate i and $m+1$ periods before the first payment date. Figure 17.1 shows the time diagram for this case where "?" indicates the present value to be found.

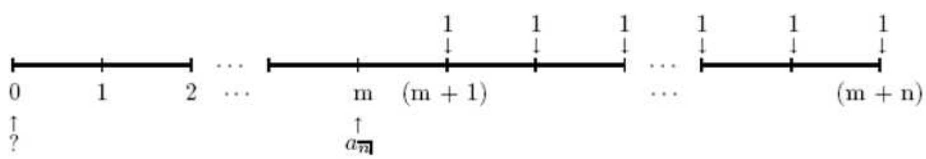


Figure 17.1

The present value of an $n-$period annuity−immediate $m+1$ periods before the first payment date (called a **deferred annuity** since payments do not begin until some later period) is the present value at time m discounted for m time periods, that is, $v^m a_{\overline{n}|}$. It is possible to express this answer strictly in terms of annuity values. Indeed,

$$a_{\overline{m+n}|} - a_{\overline{m}|} = \frac{1-\nu^{m+n}}{i} - \frac{1-\nu^m}{i} = \frac{\nu^m - \nu^{m+n}}{i} = \nu^m \frac{1-\nu^n}{i} = \nu^m a_{\overline{n}|}.$$

Such an expression is convenient for calculation, if interest tables are being used.

Example 17.1
Exactly 3 years from now is the first of four \$200 yearly payments for an annuity-immediate, with an effective 8% rate of interest. Find the present value of the annuity.

Solution.
The answer is $200\nu^2 a_{\overline{4}|} = 200(a_{\overline{6}|} - a_{\overline{2}|}) = 200(4.6229 - 1.7833) = \$567.92$ ■

The deferred−annuity introduced above uses annuity−immediate. It is possible to work with a deferred annuity−due. In this case, one can easily see that the present value is given by

$$\nu^m \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}.$$

Example 17.2
Calculate the present value of an annuity−due paying annual payments of 1200 for 12 years with the first payment two years from now. The annual effective interest rate is 6%.

Solution.
The answer is $1200(1.06)^{-2}\ddot{a}_{\overline{12}|} = 1200(\ddot{a}_{\overline{14}|} - \ddot{a}_{\overline{2}|}) = 1200(9.8527 - 1.9434) \approx 9,491.16$ ■

(2) Accumulated values more than 1 period after the last payment date
Consider the question of finding the accumulated value of an annuity−immediate with periodic interest rate i and m periods after the last payment date. Figure 17.2 shows the time diagram for this case where "?" indicates the sought accumulated value.

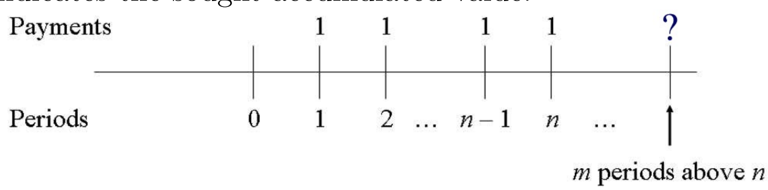


Figure 17.2

The accumulated value of an $n-$period annuity−immediate m periods after the last payment date is the accumulated value at time n accumulated for m time periods, that is, $(1+i)^m s_{\overline{n}|}$. Notice that

$$\begin{aligned} s_{\overline{m+n}|} - s_{\overline{m}|} =& \frac{(1+i)^{m+n}-1}{i} - \frac{(1+i)^m - 1}{i} \\ =& \frac{(1+i)^{m+n} - (1+i)^m}{i} = (1+i)^m \frac{(1+i)^n - 1}{i} = (1+i)^m s_{\overline{n}|} \end{aligned}$$

Example 17.3
For four years, an annuity pays \$200 at the end of each year with an effective 8% rate of interest. Find the accumulated value of the annuity 3 years after the last payment.

Solution.
The answer is $200(1+0.08)^3 s_{\overline{4}|} = 200(s_{\overline{7}|} - s_{\overline{3}|}) = 200(8.9228 - 3.2464) = \$1135.28$ ■

It is also possible to work with annuities−due instead of annuities−immediate. The reader should verify that

$$(1+i)^m \ddot{s}_{\overline{n}|} = \ddot{s}_{\overline{m+n}|} - \ddot{s}_{\overline{m}|}.$$

Example 17.4
A monthly annuity−due pays 100 per month for 12 months. Calculate the accumulated value 24 months after the first payment using a nominal rate of 4% compounded monthly.

Solution.
The answer is $100\left(1+\frac{0.04}{12}\right)^{12}\ddot{s}_{\overline{12}|\frac{0.04}{12}} = 1,276.28$ ■

(3) **Current value between the first and last payment date**
Next, we consider the question of finding the present value of an $n-$period annuity−immediate after the payment at the end of mth period where $1 \le m \le n$. Figure 17.3 shows the time diagram for this case.

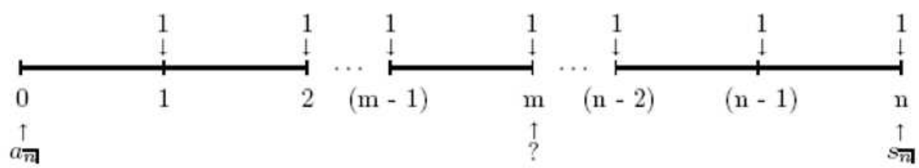


Figure 17.3

The current value of an $n-$period annuity−immediate immediately upon the mth payment date is the present value at time 0 accumulated for m time periods which is equal to the accumulated value at time n discounted for $n-m$ time periods, that is,

$$(1+i)^m a_{\overline{n}|} = \nu^{n-m} s_{\overline{n}|}.$$

One has the following formula,

$$(1+i)^m a_{\overline{n}|} = \nu^{n-m} s_{\overline{n}|} = s_{\overline{m}|} + a_{\overline{n-m}|}.$$

To see this,

$$\begin{aligned}(1+i)^m a_{\overline{n}|} =&(1+i)^m \cdot \frac{1-(1+i)^{-n}}{i}\\ =&\frac{(1+i)^m-(1+i)^{m-n}}{i}\\ =&\frac{(1+i)^m-1}{i}+\frac{1-(1+i)^{m-n}}{i}\\ =&s_{\overline{m}|}+a_{\overline{n-m}|}.\end{aligned}$$

Example 17.5
For four years, an annuity pays \$200 at the end of each half−year with an 8% rate of interest convertible semiannually. Find the current value of the annuity immediately upon the 5th payment (i.e., middle of year 3).

Solution.
The answer is $200(1.04)^5 a_{\overline{8}|0.04} = 200(s_{\overline{5}|0.04} + a_{\overline{3}|0.04}) = 200(5.4163 + 2.7751) = \$1,638.28$ ■

For annuity−due we have a similar formula for the current value

$$(1+i)^m \ddot{a}_{\overline{n}|} = \nu^{n-m} \ddot{s}_{\overline{n}|} = \ddot{s}_{\overline{m}|} + \ddot{a}_{\overline{n-m}|}.$$

Example 17.6
Calculate the current value at the end of 2 years of a 10 year annuity due of \$100 per year using a discount rate of 6%.

Solution.
We have $(1+i)^{-1} = 1 - d = 1 - 0.06 = 0.94$ and $i = \frac{0.06}{0.94}$. Thus, $100(.94)^{-2}\ddot{a}_{\overline{10}|} = \$870.27$ ■

Up to this point, we have assumed that the date is an integral number of periods. In the case the date is not an integral number of periods from each payment date, the value of an annuity is found by finding the value on a date which is an integral number of periods from each payment date and then the value on this date is either accumulated or discounted for the fractional period to the actual evaluation date. We illustrate this situation in the next example.

Example 17.7
An annuity−immediate pays \$1000 every six months for three years. Calculate the present value of this annuity two months before the first payment using a nominal interest rate of 12% compounded semiannually.

Solution.
The present value at time $t = 0$ is

$$1000 a_{\overline{6}|0.06} = 1000 \frac{1 - (1.06)^{-6}}{0.06} = \$4917.32.$$

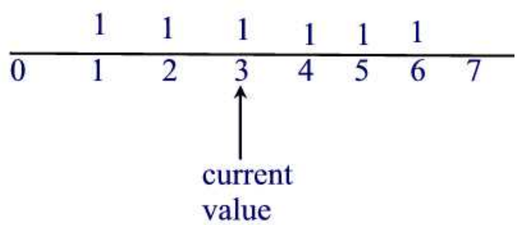
Let j be the interest rate per 2-month. Then $1 + j = (1 + 0.06)^{\frac{1}{3}}$. The present value two months before the first payment is made is

$$4917.32(1.06)^{\frac{2}{3}} = \$5112.10 \blacksquare$$

Practice Problems

Problem 17.1
For four years, an annuity pays \$200 at the end of each half-year with an 8% rate of interest convertible semiannually. Find the current value of the annuity three months after the 5th payment (i.e., nine months into year 3).

Problem 17.2
Which of the following is equal to the current value shown on the time diagram?



(i) $s_{\overline{3}|} + a_{\overline{4}|}$
(ii) $\ddot{s}_{\overline{3}|} + a_{\overline{4}|}$
(iii) $\ddot{a}_{\overline{6}|}(1+i)^3$
(iv) $v^4 s_{\overline{6}|}$
(v) $\ddot{s}_{\overline{2}|} + (1+i)a_{\overline{4}|}$

Problem 17.3
Calculate the present value of an annuity immediate with 20 annual payments of 500 if the first payment of the annuity immediate starts at the end of the fifth year. The annual effective interest rate is 8%.

Problem 17.4
Calculate the current value at the end of 5 years of an annuity due paying annual payments of 1200 for 12 years. The annual effective interest rate is 6%.

Problem 17.5
A monthly annuity due pays 100 per month for 12 months. Calculate the accumulated value 12 months after the last payment using a nominal rate of 4% compounded monthly.

Problem 17.6
A monthly annuity immediate pays 100 per month for 12 months. Calculate the accumulated value 12 months after the last payment using a nominal rate of 4% compounded monthly.

Problem 17.7
Annuities X and Y provide the following payments:

End of Year	Annuity X	Annuity Y
1 - 10	1	K
11 - 20	2	0
21 - 30	1	K

Annuities X and Y have equal present values and an annual effective interest rate of i such that $v^{10} = 0.5$. Determine K.

Problem 17.8
Payments of \$100 per quarter are made from June 7, Z through December 7, $Z + 11$, inclusive. If the nominal interest convertible quarterly is 6%:
(a) Find the present value on September 7, $Z - 1$.
(b) Find the current value on March 7, $Z + 8$.
(c) Find the accumulated value on June 7, $Z + 12$.

Problem 17.9
Find the current value to the nearest dollar on January 1 of an annuity which pays \$2,000 every six months for five years. The first payment is due on the next April 1 and the rate of interest is 9% convertible semiannually.

Problem 17.10
John buys a series of payments. The first payment of 50 is in six years. Annual payments of 50 are made thereafter until 14 total payments have been made.
Calculate the price John should pay now to realize an annual effective return of 7%.

Problem 17.11
Which of the following are true?
(i) $\ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{3}|} = a_{\overline{9}|} - a_{\overline{2}|}$
(ii) $\nu^3\ddot{a}_{\overline{3}|} = \nu^2 a_{\overline{3}|}$
(iii) $\nu^8 s_{\overline{12}|} = \ddot{a}_{\overline{3}|} + \ddot{s}_{\overline{9}|}$

Problem 17.12
A loan of 1,000 is to be repaid by annual payments of 100 to commence at the end of the fifth year and to continue thereafter for as long as necessary. The effective rate of discount is 5%. Find the amount of the final payment if it is to be larger than the regular payments.

Problem 17.13
Using an annual effective interest rate $i > 0$, you are given:
(i) The present value of 2 at the end of each year for $2n$ years, plus an additional 1 at the end of each of the first n years, is 36.
(ii) The present value of an $n-$year deferred annuity-immediate paying 2 per year for n years is 6.
Calculate i.

Problem 17.14
Show that $\sum_{t=10}^{15}(\ddot{s}_{\overline{t}|} - s_{\overline{t}|}) = s_{\overline{16}|} - s_{\overline{10}|} - 6$.

Problem 17.15
It is known that $\frac{a_{\overline{7}|}}{a_{\overline{11}|}} = \frac{a_{\overline{3}|}+s_{\overline{x}|}}{a_{\overline{y}|}+s_{\overline{z}|}}$. Find x, y, and z.

Problem 17.16
Simplify $a_{\overline{15}|}(1 + \nu^{15} + \nu^{30})$ to one symbol.

Problem 17.17
The present value of an annuity-immediate which pays \$200 every 6 months during the next 10 years and \$100 every 6 months during the following 10 years is \$4,000.
The present value of a 10-year deferred annuity-immediate which pays \$250 every 6 months for 10 years is \$2,500.
Find the present value of an annuity-immediate which pays \$200 every 6 months during the next 10 years and \$300 every 6 months during the following 10 years.

Problem 17.18 ‡
At an annual effective interest rate of $i, i > 0$, both of the following annuities have a present value of X :
(i) a 20-year annuity-immediate with annual payments of 55;
(ii) a 30-year annuity-immediate with annual payments that pays 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years.
Calculate X.

Problem 17.19
Tom borrows 100 at an annual effective interest rate of 4% and agrees to repay it with 30 annual installments. The amount of each payment in the last 20 years is set at twice that in the first 10 years. At the end of 10 years, Tom has the option to repay the entire loan with a final payment X, in addition to the regular payment. This will yield the lender an annual effective rate of 4.5% over the 10 year period. Calculate X.

Problem 17.20
You are given an annuity-immediate with 11 annual payments of 100 and a final larger payment at the end of 12 years. At an annual effective interest rate of 3.5%, the present value at time 0 of all payments is 1000.
Using an annual effective interest rate of 1%, calculate the present value at the beginning of the ninth year of all remaining payments. Round your answer to the nearest integer.

Problem 17.21
The proceeds of a 10,000 death benefit are left on deposit with an insurance company for seven years at an effective annual interest rate of 5%.
The balance at the end of seven years is paid to the beneficiary in 120 equal monthly payments of X, with the first payment made immediately. During the payout period, interest is credited at an annual effective rate of 3%.

Problem 17.22
The present value today of a 20 year annuity-immediate paying 500 every 6 months but with the first payment deferred t years has a present value of 5,805.74. If $i^{(12)} = .09$ is the interest rate used to calculate the present value, find t.

Problem 17.23
John buys an enterprise that sells dental equipment in the amount of 60 million dollars. He decides to finance the purchase by making 20 semiannual payments with the first payment in two years. Find the value of each payment if the nominal interest rate is 12% compounded semiannually.

Problem 17.24
Peter wants to accumulate \$20,000 in five years by making monthly payment at the end of each month for the first three years into a savings account that pays annual interest rate of 9% compounded monthly and then leave the accumulated amount in the account for the remaining two years. What is the value of his regular monthly payment?

18 Annuities with Infinite Payments: Perpetuities

A **perpetuity** is an annuity whose term is infinite, i.e., an annuity whose payments continue forever with the first payment occurs either immediately (perpetuity−due) or one period from now (perpetuity−immediate). Thus, accumulated values of perpetuities do not exist.
Let us determine the present value of a perpetuity−immediate at the time one period before the first payment, where a payment of 1 is made at the end of each period. The present value will be denoted by $a_{\overline{\infty}|}$. A time diagram of this case is given in Figure 18.1

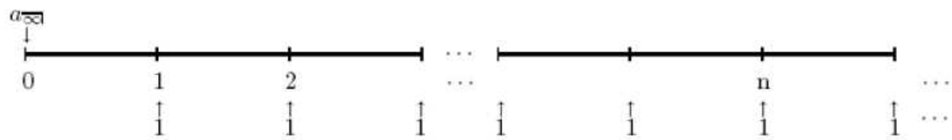


Figure 18.1

Using the equation of value at time $t = 0$ we find

$$\begin{aligned} a_{\overline{\infty}|} =& \nu + \nu^2 + \cdots \\ =& \text{infinite geometric progression with } \nu < 1 \\ =& \frac{\nu}{1-\nu} = \frac{\nu}{i\nu} = \frac{1}{i}. \end{aligned}$$

The verbal interpretation of this formula is as follows: If the periodic effective rate of interest is i then one can invest a principal of $\frac{1}{i}$ for one period and obtain a balance of $1 + \frac{1}{i}$ at the end of the first period $1. A payment of $1 is made and the remaining balance of $\frac{1}{i}$ is reinvested for the next period. This process continues forever.
Now, since $a_{\overline{n}|} = \frac{1-\nu^n}{i}$ and $\lim_{n\to\infty} \nu^n = 0$ for $0 < \nu < 1$ we have

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \frac{1}{i}.$$

Example 18.1
Suppose a company issues a stock that pays a dividend at the end of each year of $10 indefinitely, and the the companies cost of capital is 6%. What is the the value of the stock at the beginning of the year?

Solution.
The answer is $10 \cdot a_{\overline{\infty}|} = 10 \cdot \frac{1}{0.06} = \$166.67$ ■

Analogously to perpetuity−immediate, we may define a perpetuity−due to be an infinite sequence of equal payments where each payment is made at the beginning of the period. Let $\ddot{a}_{\overline{\infty}|}$ denote the present value of a perpetuity−due at the time of first payment is made. A time diagram describing this case is given in Figure 18.2.

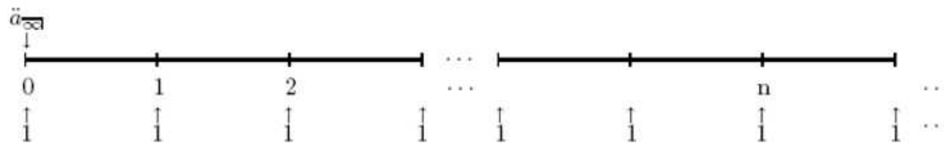


Figure 18.2

The equation of value at time $t = 0$ is

$$\ddot{a}_{\overline{\infty}|} = 1 + \nu + \nu^2 + \cdots = \frac{1}{1-\nu} = \frac{1}{d} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|}.$$

This formula can be obtained from finding the present value of a payment of \$1 at the beginning of the first period and an annutiy−immediate. Indeed,

$$1 + a_{\overline{\infty}|} = 1 + \frac{1}{i} = \frac{1+i}{i} = \frac{1}{d} = \ddot{a}_{\overline{\infty}|}.$$

Example 18.2
What would you be willing to pay for an infinite stream of \$37 annual payments (cash inflows) beginning now if the interest rate is 8% per annum?

Solution.
The answer is $37\ddot{a}_{\overline{\infty}|} = \frac{37}{0.08(1.08)^{-1}} = \$499.50$ ■

Remark 18.1
We pointed out at the beginning of the section that accumulated values for perpetuities do not exist since payments continue forever. We can argue mathematically as follows: If a perpetuity−immediate has an accumulated value, denoted by $s_{\overline{\infty}|}$, then we expect to have $s_{\overline{\infty}|} = \lim_{n\to\infty} s_{\overline{n}|}$. But

$$\lim_{n\to\infty} s_{\overline{n}|} = \lim_{n\to\infty} \frac{(1+i)^n - 1}{i}$$

does not exist since $1 + i > 1$ and $\lim_{n\to\infty}(1+i)^n = \infty$.

Perpetuities are useful in providing verbal explanations of identities. For example, the formula

$$a_{\overline{n}|} = \frac{1-\nu^n}{i} = \frac{1}{i} - \frac{\nu^n}{i} = a_{\overline{\infty}|} - \nu^n a_{\overline{\infty}|}.$$

can be interpreted as the difference between payments for two perpetuities each paying 1 at the end of each period; the first payment of the first perpetuity is one period from now and the first payment of the second perpetuity is $n+1$ periods from now. The present value of the first perpetuity is $\frac{1}{i}$ and that of the second perpetuity is $\frac{\nu^n}{i}$.
An identical relationship holds for perpetuity due. That is

$$\ddot{a}_{\overline{n}|} = \frac{1-\nu^n}{d} = \frac{1}{d} - \frac{\nu^n}{d} = \ddot{a}_{\overline{\infty}|} - \nu^n \ddot{a}_{\overline{\infty}|}.$$

Example 18.3
You can receive one of the following two sets of cash flows. Under Option A, you will receive \$5,000 at the end of each of the next 10 years. Under Option B, you will receive X at the beginning of each year, forever. The annual effective rate of interest is 10%. Find the value of X such that you are indifferent between these two options.

Solution.
The equation of value at time $t = 0$ is

$$\begin{aligned} 5000a_{\overline{10}|} =& \frac{X}{d} = X\left(\frac{1}{i}+1\right) \\ 30722.84 =& 11X \\ X =& \$2,792.99 \blacksquare \end{aligned}$$

Similar to deferred annuities, one can discuss deferred perpetuities. The present value P_0 of a deferred perpetuity−immediate with periodic payment of 1 that starts in n periods time, with a first cash flow at the beginning of period $n+1$, is given by the equation of value at time $t = n$

$$(1+i)^n P_0 = a_{\overline{\infty}|}.$$

Example 18.4
Fifty dollars is paid at the end of each year forever starting six years from now. Assume the annual effective rate of interest is 10%, find the present value of the investment.

Solution.
The deferred period is $t = 5$. The answer is

$$50(1+i)^{-5}a_{\overline{\infty}|} = 50(1.1)^{-5} \cdot \frac{1}{0.1} = \$310.46 \blacksquare$$

Example 18.5
A deferred perpetuity-immediate paying $2000 monthly is bought for a sum of $229,433.67. Find the deferred period if the interest rate compounded monthly is 6%.

Solution.
The monthly interest rate is $\frac{0.06}{12} = 0.005$. Let n be the deferred period, i.e. the time it takes $229,433.67 to accumulate to $2000a_{\overline{\infty}|}$. Thus, n satisfies the equation

$$2000a_{\overline{\infty}|} = 229,433.67(1.005)^n \Rightarrow 2000 \cdot \frac{1}{0.005} = 229,433.67(1.005)^n.$$

Solving this equation for n we find $n = 111.45$ months ■

Practice Problems

Problem 18.1
What would you be willing to pay for an infinite stream of $37 annual payments (cash inflows) beginning one year from today if the interest rate is 8%?

Problem 18.2
Which of the following are equal to $a_{\overline{3}|}$.
(i) $\nu + \nu^2 + \nu^3$
(ii) $\ddot{a}_{\overline{4}|} - 1$
(iii) $a_{\overline{\infty}|}(1 - \nu^3)$

Problem 18.3
An annuity−due pays 100 at the beginning of each year forever. The effective annual rate of interest is 25%. What is the present value of the annuity?

Problem 18.4
A special perpetuity pays $200 at the end of each year for the first 10 years and $100 at the end of each year thereafter. If the effective annual rate of interest is 10%, find the present value of the perpetuity.

Problem 18.5 ‡
A perpetuity−immediate pays X per year. Brian receives the first n payments, Colleen receives the next n payments, and Jeff receives the remaining payments. Brian's share of the present value of the original perpetuity is 40%, and Jeff's share is K. Calculate K.

Problem 18.6
A sum P is used to buy a deferred perpetuity−due of 1 payable annually. The annual effective interest rate is $i > 0$. Find an expression for the deferred period.

Problem 18.7
Deposits of $1,000 are placed in a fund at the beginning of each year for the next 20 years. After 30 years annual payments commence and continue forever, with the first payment at the end of the 30th year. Find an expression for the amount of each payment.

Problem 18.8
A benefactor leaves an inheritance to four charities, A, B, C, and D. The total inheritance is a series of level payments at the end of each year forever. During the first n years A, B, and C share each payment equally. All payments after n years revert to D. If the present values of the shares of A, B, C, and D are all equal, find $(1 + i)^n$.

Problem 18.9
A level perpetuity−immediate is to be shared by A, B, C, and D. A receives the first n payments. B the second n payments, C the third n payments, and D all payments thereafter. It is known that the ratio of the present value of C's share to A's share is 0.49. Find the ratio of the present value of B's share to D's share.

Problem 18.10
Adam buys a perpetuity due of 1,000 per month for 100,000. Calculate the annual effective rate of interest used to calculate the price of this perpetuity.

Problem 18.11
The present value of a perpetuity immediate where the payment is P is 1,000 less than the present value of a perpetuity due where the payment is P. Calculate P.

Problem 18.12
A trust has been established such that RJ will receive a perpetuity of 1000 a year with the first payment at the end of 5 years.
Calculate the present value of the perpetuity at a discount rate of $d = 8\%$.

Problem 18.13
Julie, Chris, and Allen will share an annual perpetuity immediate of 1200. Julie will receive the first 9 payments. Chris will receive the next 16 payments. Allen will receive all remaining payments.
At an annual effective interest rate of 5%, order the value of each person's share of the perpetuity.

Problem 18.14
John is receiving annual payments from a perpetuity immediate of 12,000. Krista is receiving annual payments from a perpetuity due of 10,000. If the present value of each perpetuity is equal, calculate i.

Problem 18.15
The present value of an annual perpetuity immediate of 150 is equal to the present value of an annual perpetuity immediate that pays 100 at the end of the first 20 years and 200 at the end of year 21 and each year thereafter. Calculate i.

Problem 18.16 ‡
An estate provides a perpetuity with payments of X at the end of each year. Seth, Susan, and Lori share the perpetuity such that Seth receives the payments of X for the first n years and Susan receives the payments of X for the next m years, after which Lori receives all the remaining payments of X. Which of the following represents the difference between the present value of Seth's

and Susan's payments using a constant rate of interest?
(a) $X[a_{\overline{n}|} - \nu^n a_{\overline{m}|}]$
(b) $X[\ddot{a}_{\overline{n}|} - \nu^n \ddot{a}_{\overline{m}|}]$
(c) $X[a_{\overline{n}|} - \nu^{n+1} a_{\overline{m}|}]$
(d) $X[a_{\overline{n}|} - \nu^{n-1} a_{\overline{m}|}]$
(e) $X[va_{\overline{n}|} - \nu^{n+1} a_{\overline{m}|}]$

Problem 18.17 ‡
Which of the following are characteristics of all perpetuities?(including non−level payments)
I. The present value is equal to the first payment divided by the annual effective interest rate.
II. Payments continue forever.
III. Each payment is equal to the interest earned on the principal.

Problem 18.18 ‡
A perpetuity paying 1 at the beginning of each 6-month period has a present value of 20 . A second perpetuity pays X at the beginning of every 2 years. Assuming the same annual effective interest rate, the two present values are equal. Determine X.

Problem 18.19
What is the present value of receiving \$30 at the end of each year forever, starting 9 years from now? Assume an annual rate of interest of 7%

Problem 18.20
Jim buys a perpetuity of 100 per year, with the first payment 1 year from now. The price for the perpetuity is 975.61, based on a nominal yield of i compounded semiannually. Immediately after the second payment is received, the perpetuity is sold for 1642.04, to earn for the buyer a nominal yield of j compounded semiannually. Calculate $i - j$.

Problem 18.21
Victor wants to purchase a perpetuity paying 100 per year with the first payment due at the end of year 11. He can purchase it by either
(1) paying 90 per year at the end of each year for 10 years, or
(2) paying K per year at the end of each year for the first 5 years and nothing for the next 5 years.
Calculate K.

Problem 18.22
A perpetuity pays 1 at the end of every year plus an additional 1 at the end of every second year. The present value of the perpetuity is K for $i \geq 0$. Determine K.

Problem 18.23

The accumulated value just after the last payment under a 12-year annuity-immediate of 1000 per year, paying interest at the rate of 5% per annum effective, is to be used to purchase a perpetuity at an interest rate of 6%, with first payment to be made 1 year after the last payment under the annuity. Determine the size of the payments under the perpetuity.

19 Solving for the Unknown Number of Payments of an Annuity

In this section we consider the question of finding the number of payments n given the regular payment R, the interest per period i and either the present value or the accumulated value of an annuity. We will assume an annuity−immediate. A similar calculation applies for annuity−due.
Let P be the present value of an annuity−immediate. A time diagram is given in Figure 19.1

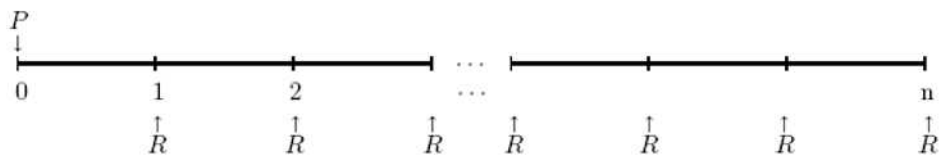


Figure 19.1

The equation of value at time $t = 0$ is $P = Ra_{\overline{n}|i}$, where P, R, and i are known quantities and n is the unknown. Solving this equation for n we find

$$P = Ra_{\overline{n}|i} = R\left(\frac{1-\nu^n}{i}\right) \Rightarrow \nu^n = 1 - i\left(\frac{P}{R}\right) \Rightarrow n = \frac{\ln\left[1 - i\left(\frac{P}{R}\right)\right]}{\ln \nu}.$$

This last expression is not necessary a positive integer. Thus, the equation $P = Ra_{\overline{n}|}$ is replaced by $P = Ra_{\overline{n+k}|}$ where n is a positive integer and $0 < k < 1$. In this case, using the results of Problems 15.47-15.48 we can write

$$P = Ra_{\overline{n+k}|i} = R\left(\frac{1-\nu^{n+k}}{i}\right) \Rightarrow \nu^{n+k} = 1 - i\left(\frac{P}{R}\right)$$

which implies

$$n + k = \frac{\ln\left[1 - i\left(\frac{P}{R}\right)\right]}{\ln \nu}.$$

It follows that, for the annuity to have the present value P, n regular payments of R must be made and an additional one payment in the amount of

$$R\left[\frac{(1+i)^k - 1}{i}\right]$$

to be made at time $t = n + k$, that is, at the fractional period k of the $(n+1)$th period.
In practice, the last smaller payment is made either at the same time as the last regular payment making the last payment larger than the regular payment (such a payment is called a **balloon**

payment) or at the end of the period following the last regular payment. In this case the smaller payment is called **drop payment.**
The following examples illustrate how to deal with the question of finding the unknown number of payments.

Example 19.1
An investment of $80,000 is to be used to make payments of $5000at the end of every six months for as long as possible. If the fund earns a nominal interest rate of 12% compounded semiannually, find how many regular payments can be made and find the amount of the smaller payment:
(a) to be paid along the last regular payment,
(b) to be paid six months after the last regular payment,
(c) to be paid during the six months following the last regular payment.

Solution.
We first solve the equation $8000 = 5000a_{\overline{n+k}|0.06}$ as follows

$$80000 = 5000a_{\overline{n+k}|0.06} = 5000\left[\frac{1-(1.06)^{-(n+k)}}{i}\right] \Rightarrow 1-(1.06)^{-(n+k)} = \frac{80000}{5000}(0.06) = 0.96$$

from which we find

$$1 - 0.96 = 0.04 = (1.06)^{-(n+k)} \Rightarrow n+k = -\frac{\ln 0.04}{\ln 1.06} = 55.242.$$

Thus, $n = 55$ and $k = 0.242$.
(a) Let X be the amount of the smaller payment to be be made at the end of the 55th period. A time diagram of this situation is given in Figure 19.2.

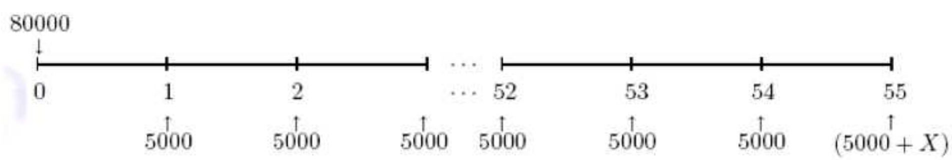


Figure 19.2

Note that every period on the time diagram consists of six months. The equation of value at time $t = 0$ is

$$80000 = 5000a_{\overline{55}|0.06} + X(1.06)^{-55} \Rightarrow 80000 = 79952.715 + (0.0405674)X \Rightarrow X = \$1165.59.$$

Thus, in this case, we have 54 payments of $5000 each, and a last payment of $5000 + 1165.59 = 6165.59$

(b) This situation is illustrated in Figure 19.3.

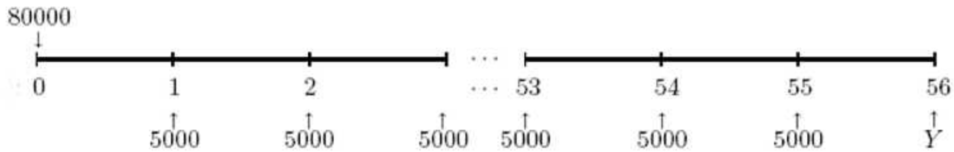


Figure 19.3

The equation of value at time $t = 0$ is

$$80000 = 5000a_{\overline{55}|0.06} + Y(1.06)^{-56} \Rightarrow 80000 = 79952.715 + (0.0382712)Y \Rightarrow Y = \$1235.53.$$

In this case, we have 55 payments of \$5000 each and a last payment of \$1235.53 six months after the last regular payment.
(c) In this case we have 55 payments of \$5000 each and one last payment of

$$5000\left[\frac{(1+0.06)^{0.242}-1}{0.06}\right] = \$1182.13$$

to be made 44 days ($0.242 \times \frac{365}{2} = 44.165$) after the last regular payment ■

A similar type of calculation can be done with known accumulated value instead of present value as illustrated in the next example.

Example 19.2
In order to accumulate \$5,000, you will deposit \$100 at the end of each month for as long as necessary. Interest is 15% compounded monthly.
(a) Find the number of regular payments and the fractional period that are required to accumulate the \$5,000?
(b) If a final fractional payment will be added to the last regular payment, what must this fractional payment be?
(c) If a final fractional payment will be made one month after the last regular payment, what must this fractional payment be?
(d) If the fractional payment to be made during the month following the last regular payment.

Solution.
(a) The monthly interest rate is $\frac{0.15}{12} = 0.0125$. The equation of value at time $t = n + k$ is

$$5000 = 100s_{\overline{n+k}|0.0125} = 100 \cdot \frac{(1+i)^{n+k}-1}{i} = 100 \cdot \frac{(1.0125)^{n+k}-1}{0.0125}$$

Solving for $n+k$ we find

$$n+k=\frac{\ln\left(1+\frac{5000(0.0125)}{100}\right)}{\ln 1.0125}=39.08$$

Thus, there will be 39 regular payments and a payment at the fractional time $0.08 = 0.08\times\frac{365}{12} = 2.433$, i.e., two days after the last regular payment.
(b) Let X be the fractional payment added to the last regular payment. The time diagram of this case is shown in Figure 19.4.

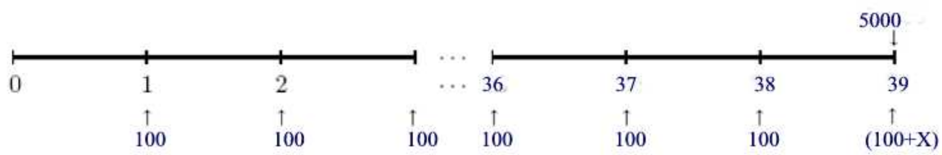


Figure 19.4

The equation of value at time $t=39$ is $5000 = 100s_{\overline{39}|0.0125} + X$. Solving for x we find $X = \$13.38$.
(c) Let Y be such a payment. The time diagram of this case is shown in Figure 19.5.

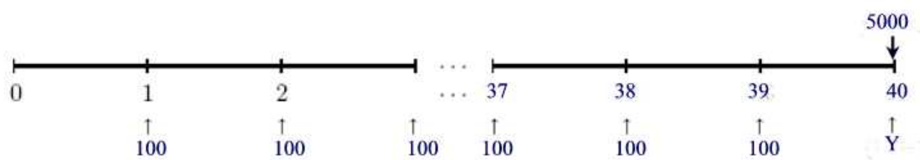


Figure 19.5

The equation of value at time $t=40$ is

$$5000 = 100\ s_{\overline{39}|}(1.0125) + Y.$$

Solving for Y we find $Y = -48.96$.
What does this negative answer mean? If 39 regular payments are made at the end of each month, then at time 39 (measured in months) the account will have accumulated $100s_{\overline{39}|0.0125} = 4986.62$. This is extremely close to the 5000 that we wanted in the first place. If no additional payment is made from time 39 to time 40, then interest alone will make the account have $4986.62(1.0125) = 5048.96$. This means that there will be no additional payment required at time 40. In fact, \$48.96 would have to be withdrawn from the account to make the account have only \$5000 at time 40.
(d) In tis case, 39 regular payments of \$100 each to be made and one additional payment in the amount of

$$100\left[\frac{(1+0.0125)^{0.08}-1}{0.0125}\right] = \$7.95$$

to be made 2 days after the last regular payment ■

Example 19.3
An investment of $10,000 is to be used to make payments of $700 at the end of every year for as long as possible. If the effective annual rate if interest is 6%, find:
(a) the number of regular payments the fund will make,
(b) the size of the smaller final payment if the payment will be made in addition to the last regular payment,
(c) the size of the smaller final payment if the payment will be made one year after the last regular payment.
(d) the size of the smaller final payment if the payment is to be made during the year following the last regular payment.

Solution.
(a) We have $10000 = 700a_{\overline{n}|}$ or $10000 = 700 \cdot \frac{1-(1+0.06)^{-n}}{0.06}$. Solving for n we find

$$n = \frac{\ln\left(1 - \frac{10000(0.06)}{700}\right)}{-\ln 1.06} = 33.40$$

Thus, there will be 33 regular payments.
(b) Let x be the size of the payment. The equation of value at $t = 0$ is

$$10000 = 700a_{\overline{33}|} + v^{33}x$$

or

$$10000(1.06)^{33} = 700s_{\overline{33}|} + x.$$

Solving for x we find $x = 265.68$.
(c) Let x be the size of the payment. The equation of value at time $t = 0$ is

$$10000 = 700a_{\overline{33}|} + v^{34}x.$$

Solving for x we find $x = 281.62$.
(d) The equation of value at time $t = 0$ is

$$700a_{\overline{33+k}|} = 10000$$

or

$$0.07a_{\overline{33+k}|} = 1$$

where $0 < k < 1$. (See Problem 15.29). Thus, we have $0.07\frac{1-v^{33+k}}{i} = 1 \Rightarrow v^{33+k} = 1 - \frac{6}{7} = \frac{1}{7} \Rightarrow 33 + k = \frac{\ln 7}{\ln 1.06} \approx 33.395 \Rightarrow k = 0.395$. In this case, the equation of value at time $t = 0$ is

$$700a_{\overline{33}|} + (1.06)^{-33.395}x = 10000.$$

Solving for x we find $x = \$271.87$. That is, the final irregular payment of $271.87 is paid at time 33.395 ■

Practice Problems

Problem 19.1
Allison pays \$27,506.28 for an annuity due today. This annuity will pay her \$1,750 at the beginning of each month for n months with the first payment coming today. If $i^{(12)} = 12\%$, find n. Round your answer to the nearest integer.

Problem 19.2
In order to accumulate \$2,000, you will deposit \$100 at the end of each month for as long as necessary. Interest is $i^{(12)} = 4\%$.
(a) How many regular payments are required to accumulate the \$2,000?
(b) If a final fractional payment will be added to the last regular payment, what must this fractional payment be?

Problem 19.3 ‡
The present values of the following three annuities are equal:
1. perpetuity-immediate paying 1 each year, calculated at an annual effective interest rate of 7.25%;
2. 50-year annuity-immediate paying 1 each year, calculated at an annual effective interest rate of $j\%$;
3. $n-$year annuity-immediate paying 1 each year, calculated at an annual effective interest rate of $(j-1)\%$.
Calculate n.

Problem 19.4
Jenna is the beneficiary of a fund of 20,000 that pays her 1,000 at the end of each month. The fund earns 6% compounded monthly. The final payment to exhaust the fund will be a balloon payment. Calculate the amount of the balloon payment.

Problem 19.5
Jordan inherits 50,000. This inheritance is invested in a fund earning an annual rate of interest of 6%. He withdraws 5,000 per year beginning immediately. Once Jordan can no longer withdraw a full 5,000, he will withdraw a final payment one year after the prior regular payment. Calculate the final payment.

Problem 19.6
A loan of 1,000 is to be repaid by annual payments of 100 to commence at the end of the 5th year, and to continue thereafter for as long as necessary. Find the time and amount of the final payment if the final payment is to be larger than the regular payments. Assume $i = 4.5\%$.

Problem 19.7
A fund of 2,000 is to be accumulated by n annual payments of 50 by the end of each year, followed by n annual payments of 100 by the end of each year, plus a smaller final payment made 1 year after the last regular payment. If the effective rate of interest is 4.5%, find n and the amount of the final irregular payment.

Problem 19.8
One annuity pays 4 at the end of each year for 36 years. Another annuity pays 5 at the end of each year for 18 years. The present values of both annuities are equal at effective rate of interest i. If an amount of money invested at the same rate i will double in n years, find n.

Problem 19.9
A fund earning 8% effective is being accumulated with payments of 500 at the beginning of each year for 20 years. Find the maximum number of withdrawals of 1,000 which can be made at the end of each year under the condition that once withdrawals start they must continue through the end of the 20-year period.

Problem 19.10
A borrower has the following two options for repaying a loan:
(i) Sixty monthly payments of 100 at the end of each month.
(ii) A single payment of 6,000 at the end of K months.
Interest is at the nominal annual rate of 12% convertible monthly. The two options have the same present value. Find K to the nearest integer.

Problem 19.11 ‡
An annuity pays 1 at the end of each year for n years. Using an annual effective interest rate of i, the accumulated value of the annuity at time $(n+1)$ is 13.776 . It is also known that $(1+i)^n = 2.476$. Calculate n.

Problem 19.12
Ten annual deposits of 1,000 each are made to account A, starting on January 1, 1986. Annual deposits of 500 each are made to account B indefinitely, also starting on January 1, 1986. Interest on both accounts is $i = 0.05$, with interest credited on December 31.
On what date will the balance in account B be larger than the balance in account A? Assume the only transactions are deposits and interest credited to the accounts on December 31.

Problem 19.13 ‡
You are given an annuity-immediate with 11 annual payments of 100 and a final payment at the end of 12 years. At an annual effective interest rate of 3.5%, the present value at time 0 of all payments is 1,000.
Calculate the final payment.

Problem 19.14
A loan of 10,000 is being repaid with payment of 300 at the end of each month for as long as necessary plus an additional payment at the time of the last regular payment. What is the amount of the additional payment using an interest rate of 9% compounded monthly.

Problem 19.15
A loan of 10,000 is being repaid with payments of 500 starting one month after the loan is made and lasting as long as necessary. A final smaller payment is made one month after the last regular payment of 500.
What is the amount of the additional smaller payment using an interest rate of 12% compounded monthly?

Problem 19.16
You are given an annuity immediate with 11 annual payments of 100 and a final balloon payment at the end of 12 years. At an annual effective interest rate of 3.5%, the present value at time 0 of all the payments is 1,000. Using an annual effective interest rate of 1%, calculate the present value at the beginning of the ninth year of all remaining payments.

Problem 19.17
On the first day of every January, April, July and October Smith deposits 100 in an account earning $i^{(4)} = 0.16$. He continues the deposits until he accumulates a sufficient balance to begin withdrawals of 200 every 3 months, starting three months after the final deposit such that he can make twice as many withdrawals as he made deposits. How many deposits are needed?

Problem 19.18
On the first day of the month, starting January 1, 1995, Smith deposits 100 in an account earning $i^{(12)} = 0.09$, with interest credited the last day of each month. In addition, Smith deposits 1,000 in the account every December 31.
Around what date does the account first exceed 100,000?

20 Solving for the Unknown Rate of Interest of an Annuity

In this section we consider the problem of finding the interest rate given the number of payments and either the present value or the accumulated value of the payments. That is, we want to solve an equation like $a_{\overline{n}|} = k$ or $s_{\overline{n}|} = k$ for i.
We consider three methods in determining an unknown rate of interest. The first is to solve for i by algebraic techniques. For example, the expression

$$k = a_{\overline{n}|} = \nu + \nu^2 + \cdots + \nu^n$$

is a polynomial in ν of degree n. If the roots of this polynomial can be determined algebraically, then i is immediately determined. That is, $i = \nu^{-1} - 1$. This method is generally practical only for small values of n.

Example 20.1
If $a_{\overline{2}|} = 1.75$, find an exact expression for $i > 0$.

Solution.
We have

$$\begin{aligned} a_{\overline{2}|} = 1.75 &\Leftrightarrow \nu + \nu^2 = 1.75 \\ &\Leftrightarrow \nu^2 + \nu - 1.75 = 0 \\ &\Leftrightarrow \nu = \frac{-1 + \sqrt{8}}{2}. \end{aligned}$$

Since $(1+i)^{-1} = \frac{-1+\sqrt{8}}{2}$, solving this for i we find $i = \frac{2}{\sqrt{8}-1} - 1 \approx 0.09384$ ■

Example 20.2
Solve $s_{\overline{3}|i} = 3.31$ for i.

Solution.
We have

$$\frac{(1+i)^3 - 1}{i} = 3.31.$$

This reduces to $i^3 + 3i^2 - 0.31i = 0$ and can be factored to $i(i + 3.1)(i - 0.1) = 0$. $i = 0$ is clearly an extraneous solution and is not the correct answer. Hence, since $i > 0$ the correct answer is $i = 0.1 = 10\%$ ■

Example 20.3
The Honors Society decides to set up a scholarship for university students. They deposit \$1,000 in an account for 11 years with the first payment a year from now, and then transfer the accumulated value to a perpetuity-immediate paying \$500 each year to a deserving student. Assuming the account and the perpetuity are at the same effective annual interest rate i, find i.

Solution.
The accumulated value in the account after the last deposit is $100s_{\overline{11}|}$. The present value of the perpetuity at that time is $500a_{\overline{\infty}|}$; the accumulated value and the value of the perpetuity must be equal.

$$\begin{aligned}1000s_{\overline{11}|} =&500a_{\overline{\infty}|}\\ 2\cdot\frac{(1+i)^{11}-1}{i} =&\frac{1}{i}\\ (1+i)^{11} =&1.5\\ i =&1.5^{\frac{1}{11}}-1\approx 3.75\%\ \blacksquare\end{aligned}$$

The second method for determining the unknown rate is to use linear interpolation (see Section 13).

Example 20.4
An annuity−immediate pays you \$100 in each of the next eight years. Its present value is \$680. Using linear interpolation of the interest table, find an estimate of the annual effective rate i.

i	$s_{\overline{8}\rvert}$	$a_{\overline{8}\rvert}$
2.5%	8.7361	7.1701
3%	8.8923	7.0197
3.5%	9.0517	6.8740
4%	9.2142	6.7327
4.5%	9.3800	6.5959
5%	9.5491	6.4632
6%	9.8975	6.2098
7%	10.2598	5.9713

Solution.
From the given table we see that $a_{\overline{8}|0.035} = 6.8740$ and $a_{\overline{8}|0.04} = 6.7327$. Using linear interpolation (See Section 13), we find

$$i\approx 0.035+(0.04-0.035)\cdot\frac{6.80-6.8740}{6.7327-6.8740} = 3.762\%\ \blacksquare$$

Example 20.5
At what interest rate, convertible quarterly, is \$16,000 the present value of \$1,000 paid at the end of every quarter for five years?

Solution.
Let $j = \frac{i^{(4)}}{4}$ so that the equation of value at time $t = 0$ becomes

$$1000a_{\overline{20}|j} = 16,000$$

or

$$a_{\overline{20}|j} = 16.$$

From the tables of interest we see that $a_{\overline{20}|0.02} = 16.3514$ and $a_{\overline{20}|0.0250} = 15.5892$. Thus, using linear interpolation we find

$$j = 0.02 + (0.0250 - 0.0200) \cdot \frac{16 - 16.3514}{15.5892 - 16.3514} = 0.0223$$

which gives $i^{(4)} = 4(0.0223) = 0.0892 = 8.92\%$ ■

i	$a_{\overline{20}\vert}$
1.0%	18.0456
1.5%	17.1686
2.0%	16.3514
2.5%	15.5892

The third method is to use the *Newton-Raphson iteration method*. This method is used to approximate the zeros of the equation $f(x) = 0$ where f is a differentiable function. The idea of the method is to start with an initial guess x_0 and then find the equation of the tangent line at x_0. We next find the $x-$intercept of this line say x_1, which is closer to the real solution than x_0. See Figure 20.1. One can easily see that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Next, we find the equation of the tangent line at $x = x_1$ and find the $x-$intercept of this line. Say,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

The iteration continues so that we generate a sequence $x_0, x_1, \cdots, x_n, \cdots$ with

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

x_n converging to the solution of $f(x) = 0$.

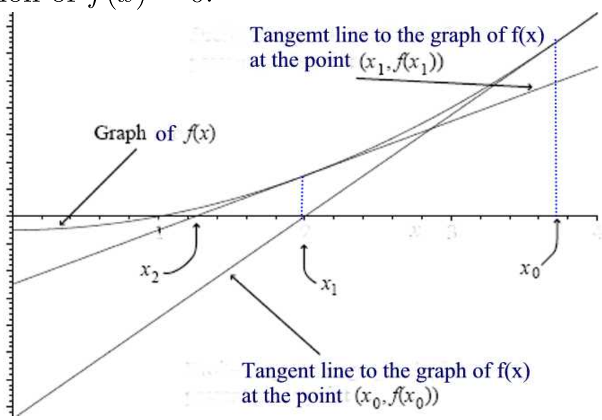


Figure 20.1

In our context, we want to solve $a_{\overline{n}|} = k$ for i using the above mentioned method. Equivalently, we want to solve the equation

$$f(i) = \frac{1-(1+i)^{-n}}{i} - k = 0.$$

The iteration formula in this case is given by

$$\begin{aligned} i_{s+1} =& i_s - \left[\frac{[1-(1+i_s)^{-n}]i_s^{-1} - k}{n(1+i_s)^{-n-1}i_s^{-1} + [1-(1+i_s)^{-n}]i_s^{-2}}\right] \\ =& i_s\left[1 + \frac{1-(1+i_s)^{-n} - ki_s}{1-(1+i_s)^{-n-1}\{1+i_s(n+1)\}}\right] \end{aligned}$$

The next question regarding this third method is the selection of the initial guess i_0. From the power series of $(1+i)^{-n}$ we find

$$\begin{aligned} a_{\overline{n}|} =& \frac{1}{i}(1-(1+i)^{-n}) \\ =& \frac{1}{i}\left(1 - \left[1 - ni + \frac{(-n)(-n-1)}{2!}i^2 + \cdots\right]\right) \\ =& n\left[1 - \frac{n+1}{2!}i + \frac{(n+1)(n+2)}{3!}i^2 - \cdots\right] \end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{a_{\overline{n}|}} =& \frac{1}{n}\left[1-\left(\frac{n+1}{2!}i-\frac{(n+1)(n+2)}{3!}i^2+\cdots\right)\right]^{-1}\\ =& \frac{1}{n}\left[1+\left(\frac{n+1}{2!}i-\frac{(n+1)(n+2)}{3!}i^2+\cdots\right)+\left(\frac{n+1}{2!}i-\frac{(n+1)(n+2)}{3!}i^2+\cdots\right)^2+\cdots\right]\\ =& \frac{1}{n}\left[1+\frac{n+1}{2!}i+\frac{n^2-1}{12}i^2-\frac{n(n^2-1)}{24}i^3+\cdots\right].\end{aligned}$$

The rate of convergence of this series is much faster than the one for the series expansion of $a_{\overline{n}|}$. Now, for the initial guess we use the approximation

$$\frac{1}{k} \approx \frac{1}{n}\left[1+\frac{n+1}{2}i_0\right].$$

Solving for i_0 we find

$$i_0 = \frac{2(n-k)}{k(n+1)}.$$

In practice, the iterations will be carried out until $i_{s+1} = i_s$ to the required degree of accuracy.

Example 20.6
Rework Example 20.5 using Newton-Raphson iterations

Solution.
Our starting value is $i_0 = \frac{2(n-k)}{k(n+1)} = \frac{2(20-16)}{16(20+1)} = 0.0238$. Finding several iterations we obtain

$$\begin{aligned}i_1 =& 0.0238\left[1+\frac{1-(1.0238)^{-20}-16(0.0238)}{1-(1.0238)^{-21}\{1+0.0238\times 21\}}\right] = 0.022246\\ i_2 =& 0.022246\left[1+\frac{1-(1.022246)^{-20}-16(0.022246)}{1-(1.022246)^{-21}\{1+0.022246\times 21\}}\right] = 0.0222623\\ i_3 =& 0.0222623\left[1+\frac{1-(1.0222623)^{-20}-16(0.0222623)}{1-(1.0222623)^{-21}\{1+0.0222623\times 21\}}\right] = 0.0222623\end{aligned}$$

Thus, a more accurate interest rate to Example 20.5 is $i^{(4)} = 4(0.0222623) = 8.9049\%$ ■
Next, consider the problem of finding i solution to the equation

$$s_{\overline{n}|i} = k.$$

For this, we let

$$f(i) = \frac{(1+i)^n - 1}{i} - k.$$

The Newton-Raphson iterations are given by

$$i_{s+1} = i_s \left[1 + \frac{(1+i_s)^n - 1 - ki_s}{(1+i_s)^{n-1}\{1 - i_s(n-1)\} - 1}\right]$$

and the initial guess is

$$i_0 \approx \frac{2(k-n)}{k(n-1)}.$$

See Problem 20.20

Practice Problems

Problem 20.1
(a) The accumulated value of a five-year annuity-immediate with semiannual payments of \$2,000 is \$25,000. Use linear interpolation of the interest table to find an estimate for i.
(b) The present value of a 11-year annuity-due with annual payments of \$2,000 is \$18,000. Use linear interpolation of the interest table to find an estimate for i.

Problem 20.2 ‡
To accumulate 8,000 at the end of $3n$ years, deposits of 98 are made at the end of each of the first n years and 196 at the end of each of the next $2n$ years. The annual effective rate of interest is i. You are given $(1+i)^n = 2.0$. Determine i.

Problem 20.3 ‡
For 10,000, Kelly purchases an annuity-immediate that pays 400 quarterly for the next 10 years. Calculate the annual nominal interest rate convertible monthly earned by Kelly's investment.

Problem 20.4 ‡
A 10-year loan of \$20,000 is to be repaid with payments at the end of each year. It can be repaid under the following two options:
(X) Equal annual payments at the annual effective rate of interest of 8%;
(Y) Installments of \$2,000 each year plus interest on the unpaid balance at an annual effective rate of i.
The sum of the payments under option (X) equals the sum of the payments under option (Y). Determine i.

Problem 20.5
Given $a_{\overline{\infty}|} = X$ and $\ddot{a}_{\overline{\infty}|} = 1.25X$. Calculate the interest rate used.

Problem 20.6
A perpetuity pays 1 at the beginning of every year. The present value is 10. Calculate the annual effective rate of interest earned by the perpetuity.

Problem 20.7
If $\ddot{a}_{\overline{2}|} \cdot s_{\overline{2}|} = 4.05$, calculate i.

Problem 20.8
A deferred perpetuity pays 500 annually beginning at the end of year 5. The present value of the deferred perpetuity is 4,992. Calculate the annual effective interest rate used to calculate the present value.

Problem 20.9
An annuity due which pays 100 per month for 12 years has a present value of 7,908.
Calculate the annual effective interest rate used to determine the present value.

Problem 20.10
An annuity which pays 200 at the end of each quarter for 5 years has a present value of 3,600.
Calculate the nominal rate of interest compounded quarterly.

Problem 20.11
An annuity immediate pays 750 per year for 15 years. The accumulated value of the annuity after 15 years is 15,000.
Calculate the annual effective rate of interest used to calculate the accumulated value.

Problem 20.12
Adam buys a perpetuity due of 1000 per month for 100,000. Calculate the annual effective rate of interest used to calculate the price of this perpetuity.

Problem 20.13
A beneficiary receives a 10,000 life insurance benefit. If the beneficiary uses the proceeds to buy a 10-year annuity-immediate, the annual payout will be 1,538. If a 20-year annuity-immediate is purchased, the annual payout will be 1,072. Both calculations are based on an annual effective interest rate of i. Find i.

Problem 20.14
If $\ddot{s}_{\overline{n}|} = 11$ and $s_{\overline{n+1}|} = 2a_{\overline{n+1}|}$, find i.

Problem 20.15
If $s_{\overline{n}|} = 10$ and $s_{\overline{2n}|} = 24$, find ν^n.

Problem 20.16 ‡
Kathryn deposits 100 into an account at the beginning of each 4-year period for 40 years. The account credits interest at an annual effective interest rate of i. The accumulated amount in the account at the end of 40 years is X, which is 5 times the accumulated amount in the account at the end of 20 years. Calculate X.

Problem 20.17
The following two investment plans result in the same accumulated amount at the end of 10 years.
1. You lend \$10,000, to be repaid in 10 equal installments at the end of each year, at the annual rate of interest of 6%. The installments are deposited to a savings account paying 3% effective annual rate of interest.
2. You deposit \$10,000 to an account paying the annual effective rate of interest i.
Find i.

Problem 20.18 ‡
At an annual effective interest rate of $i, i > 0\%$, the present value of a perpetuity paying 10 at the end of each 3-year period, with the first payment at the end of year 3, is 32.
At the same annual effective rate of i, the present value of a perpetuity paying 1 at the end of each 4-month period, with first payment at the end of 4 months, is X. Calculate X.

Problem 20.19
A fund of \$17,000 is to be accumulated at the end of five years with payments at the end of each half-year. The first five payments are \$1000 each, while the second five are \$2000 each. Find the nominal rate of interest convertible semiannually earned on the fund.

Problem 20.20
Consider solving the equation $s_{\overline{n}|} = k$.
(a) Show that $s_{\overline{n}|} = n\left(1 + \frac{n-1}{2!}i + \frac{(n-1)(n-2)}{3!}i^2 + \cdots\right)$.
(b) Show that $[s_{\overline{n}|}]^{-1} = \frac{1}{n}\left(1 - \frac{n-1}{2}i + \cdots\right)$.
(c) Show that the Newton-Raphson iterations are given by

$$i_{s+1} = i_s\left[1 + \frac{(1+i_s)^n - 1 - ki_s}{(1+i_s)^{n-1}\{1-(n-1)i_s\} - 1}\right].$$

(d) Show that an initial guess for Newton-Raphson method is given by

$$i_0 \approx \frac{2(n-k)}{k(n-1)}.$$

Problem 20.21
Use the Newton-Raphson iteration method with three iterations to estimate the rate of interest if $s_{\overline{20}|i} = 25$.

Problem 20.22
Eric receives 12000 from a life insurance policy. He uses the fund to purchase two different annuities, each costing 6000. The first annuity is a 24 year annuity immediate paying K per year to himself. The second annuity is an 8 year annuity immediate paying $2K$ per year to his son. Both annuities are based on an annual effective interest rate of $i > 0$. Determine i.

Problem 20.23
Jeff deposits 100 at the end of each year for 13 years into Fund X. Antoinette deposits 100 at the end of each year for 13 years into Fund Y. Fund X earns an annual effective rate of 15% for the first 5 years and an annual effective rate of 6% thereafter. Fund Y earns an annual effective rate of i. At the end of 13 years, the accumulated value of Fund X equals the accumulated value of Fund Y. Calculate i.

Problem 20.24

Dottie receives payments of X at the end of each year for n years. The present value of her annuity is 493. Sam receives payments of $3X$ at the end of each year for $2n$ years. The present value of his annuity is 2748. Both present values are calculated at the same annual effective interest rate. Determine ν^n.

Problem 20.25 ‡

A discount electronics store advertises the following financing arrangement:

"We don't offer you confusing interest rates. We'll just divide your total cost by 10 and you can pay us that amount each month for a year."

The first payment is due on the date of sale and the remaining eleven payments at monthly intervals thereafter.

Calculate the effective annual interest rate the store's customers are paying on their loans.

21 Varying Interest of an Annuity

In this section we consider situations in which interest can vary each period, but compound interest is still in effect.
Let i_k denote the rate of interest applicable from time $k-1$ to time k. We consider first the present value of an $n-$period annuity-immediate.
We consider the following two variations. The first is when i_k is applicable only for period k regardless of when the payment is made. That is, the rate i_k is used only in period k for discounting all payments. In this case the present value is given by

$$a_{\overline{n}|} = (1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_1)^{-1}(1+i_2)^{-1}\cdots(1+i_n)^{-1}.$$

The second variation is when the rate i_k is used as the effective rate for each period $i \le k$ when a payment is made at time k. In this case, the present value is

$$a_{\overline{n}|} = (1+i_1)^{-1} + (1+i_2)^{-2} + \cdots + (1+i_n)^{-n}.$$

Present value of annuity-due can be obtained from present value of annuity-immediate by using the formula

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}.$$

We now turn to accumulated values. We will consider an annuity-due. Again we consider two different situations. If i_k is applicable only for period k regardless of when the payment is made, then the accumulated value is given by

$$\ddot{s}_{\overline{n}|} = (1+i_1)(1+i_2)\cdots(1+i_n) + \cdots + (1+i_{n-1})(1+i_n) + (1+i_n).$$

For i_k applicable for all periods $i \ge k$, the accumulated value is

$$\ddot{s}_{\overline{n}|} = (1+i_1)^n + (1+i_2)^{n-1} + \cdots + (1+i_n).$$

Accumulated values of annuity-immediate can be obtained from the accumulated values of annuity-due using the formula

$$s_{\overline{n+1}|} = \ddot{s}_{\overline{n}|} + 1.$$

Example 21.1
Find the accumulated value of a 12-year annuity-immediate of \$500 per year, if the effective rate of interest (for all money) is 8% for the first 3 years, 6% for the following 5 years, and 4% for the last 4 years.

Solution.
The accumulated value of the first 3 payments to the end of year 3 is

$$500s_{\overline{3}|0.08} = 500(3.2464) = \$1623.20.$$

The accumulated value of the first 3 payments to the end of year 8 at 6% and then to the end of year 12 at 4% is

$$1623.20(1.06)^5(1.04)^4 = \$2541.18.$$

The accumulated value of payments 4, 5, 6, 7, and 8 at 6% to the end of year 8 is

$$500s_{\overline{5}|0.06} = 500(5.6371) = \$2818.55.$$

The accumulated value of payments 4, 5, 6, 7, and 8 to the end of year 12 at 4% is

$$2818.55(1.04)^4 = \$3297.30.$$

The accumulated value of payments 9, 10, 11, and 12 to the end of year 12 at 4% is

$$500s_{\overline{4}|0.04} = 500(4.2465) = \$2123.23.$$

The accumulated value of the 12-year annuity immediate is

$$2541.18 + 3297.30 + 2123.23 = \$7961.71 \blacksquare$$

Example 21.2
How much must a person deposit now into a special account in order to withdraw \$1,000 at the end of each year for the next fifteen years, if the effective rate of interest is equal to 7% for the first five years, and equal to 9% for the last ten years?

Solution.
The answer is

$$\begin{aligned} PV =& 1000(a_{\overline{5}|0.07} + a_{\overline{10}|0.09}(1.07)^{-5}) \\ =& 1000(4.1002 + 4.5757) = \$8675.90 \blacksquare \end{aligned}$$

Example 21.3
Find $s_{\overline{5}|}$, if $\delta_t = 0.02t$ for $0 \le t \le 5$.

Solution.
$s_{\overline{5}|}$ is equal to the sum of the accumulated value of the individual payments. That is,

$$s_{\overline{5}|} = e^{\int_1^5 \delta_t dt} + e^{\int_2^5 \delta_t dt} + \cdots + 1$$

Since

$$e^{\int_r^5 \delta_t dt} = e^{\int_r^5 0.02t dt} = e^{.25 - 0.01r^2}$$

we find

$$s_{\overline{5}|} = e^{0.24} + e^{0.21} + e^{0.16} + e^{0.09} + 1 \approx 5.7726 \blacksquare$$

Practice Problem

Problem 21.1
James deposits 1,000 into an account at the end of each year for the next 6 years. The account earns 5% interest. James also deposits 1,000 at the end of each of years 7 through 10 into another account earning 4%. Calculate the total amount James will have in both accounts at the end of ten years.

Problem 21.2
A fund earns 5% during the next six years and 4% during years 7 through 10. James deposits 1,000 into the account now. Calculate his accumulated value after 10 years.

Problem 21.3
A fund earns 5% during the next six years and 4% during years 7 through 10. James deposits 1,000 into the account at the end of each year for the next ten years. Calculate his accumulated value after 10 years.

Problem 21.4
A perpetuity pays $1,200 at the beginning of each year with the first payment being made immediately. The trust funding the perpetuity will earn an annual effective interest rate of 10% for the first 10 years, 8% for the second 10 years and 5% thereafter.
Calculate the amount needed to fund the perpetuity immediately before the first payment.

Problem 21.5
You are given:
(i) X is the current value at time 2 of a 20-year annuity-due of 1 per annum.
(ii) The annual effective interest rate for year t is $\frac{1}{8+t}$.
Find X.

Problem 21.6
Fund A pays interest of 4% on all money deposited in the first 5 years and 5% on all money deposited thereafter. Fund B pays interest of 4% during the first 5 years and 5% thereafter without regard to the date the deposit was made.
(a) If Kevin deposits 500 into each fund now, how much will he have after 10 years in each fund?
(b) Heather deposits 100 at the start of each year for 10 years into Fund B. Lisa deposits 100 at the end of each year into Fund A. Who will have more after 10 years and how much more will that person have?

Problem 21.7
A loan P is to be repaid by 10 annual payments beginning 6 months from the date of the loan. The first payment is to be half as large as the others. For the first $4\frac{1}{2}$ years interest is i effective; for the remainder of the term interest is j effective. Find an expression for the first payment.

Problem 21.8
Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the rate of interest is 8% convertible semiannually for the first three years and 7% convertible semiannually for the last two years.

Problem 21.9
Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the payments for the first three years are discounted at 8% convertible semiannually and the payments for the last two years are discounted at 7% convertible semiannually.

Problem 21.10
Given that $\delta_t = \frac{1}{20-t},\ t \geq 0$, find $s_{\overline{10}|}$.

Problem 21.11
What is the cost of an annuity of \$300 per year for 15 years where the future interest rate for the first 5 years is 4%, the rate for the next 5 years is 5%, and the rate for the last 5 years is 6%.

Problem 21.12
A perpetuity pays \$200 each year, with the first payment scheduled one year from now. Assume that the effective annual interest rate for the next five years is 8%, and thereafter it is 5%. Find the present value of the perpetuity.

22 Annuities Payable at a Different Frequency than Interest is Convertible

In this section we address annuities for which payment period and the interest conversion period differ and for which the payments are level amount. For example, an annuity-due with monthly payment of \$2,000 and with interest rate say 12% convertible quarterly.
The approach we use in this section for solving this type of problem consists of the following two steps:
(i) Find the rate of interest convertible at the same frequency as payments are made, which is equivalent to the given rate of interest. In the example above, we need to find the interest rate convertible monthly which is equivalent to the rate of 12% convertible quarterly. In this case, we find $i^{(12)} = 12\left[\left(1+\frac{0.12}{4}\right)^{\frac{4}{12}} - 1\right] = 11.9\%$.
(ii) Using this new interest rate, find the value of the annuity using the techniques discussed in the previous sections.
We illustrate this approach in the following examples.

Example 22.1
You want to accumulate \$250,000. You intend to do this by making deposits of \$2,000 into an investment account at the beginning of each month. The account earns 12% interest, convertible quarterly. How many months will it take to reach your goal? (Round to the nearest month.)

Solution.
We are given an interest rate of 12% per quarter. Let j be the equivalent rate of interest per month, which is the payment period. We have $(1+j)^{12} = \left(1+\frac{0.12}{4}\right)^4$. Solving for j we find $j = (1.03)^{\frac{1}{3}} - 1 = 0.9901634\%$. Thus, $250{,}000 = 2000\ddot{s}_{\overline{n}|j} \to \frac{(1+j)^n-1}{\frac{j}{j+1}} = 125 \to (1+j)^n = 2.22557 \to n = \frac{\ln 2.22557}{\ln 1.009901634} = 81.195 \approx 81$ months ■

Example 22.2
A loan of \$3,000 is to be repaid with quarterly installments at the end of each quarter for 5 years. If the rate of interest charged on the loan is 10% convertible semi-annually, find the amount of each quarterly payment.

Solution.
We are given an interest rate of 5% per half-year. Let j be the equivalent rate of interest per quarter, which is the payment period. We have $(1+j)^4 = (1+0.05)^2 \to j = (1.05)^{\frac{1}{2}} - 1 = 0.024695$. If R denote the quarterly payment then the equation of value at time $t = 0$ is $Ra_{\overline{20}|j} = 3000$. Solving for R we find $R = \frac{3000}{15.6342} = \$191.89$ ■

Example 22.3
The nominal rate of interest, convertible semiannually, is 6%. An annuity-immediate pays \$50 each month for five years. Find the accumulated value of this annuity at the time of the last payment.

Solution.
We are given an interest rate of 3% per half-year. Let j be the equivalent rate of interest per month, which is the payment period. We have $(1+j)^{12} = (1+0.03)^2 \to j = (1.03)^{\frac{1}{6}} - 1 = 0.49386\%$. The accumulated value is $50s_{\overline{60|}j} = 50 \cdot \frac{(1+j)^{60}-1}{j} = \$3,481.91$ ■

Example 22.4
At what annual effective rate of interest will payments of \$100 at the end of every quarter accumulate to \$2500 at the end of five years?

Solution.
Let j be the effective rate per quarter. We are given that $100s_{\overline{20|}j} = 2500$ or $(1+j)^{20} - 1 = 25j$. Let $f(j) = (1+j)^{20} - 25j - 1$. By trial and error we find $f(0.02) = -0.014053$ and $f(0.024) = 0.006938$. Using linear interpolation we find

$$j \approx 0.02 + 0.014053 \times \frac{0.024 - 0.02}{0.006938 + 0.014053} = 0.02268.$$

Let i be the annual effective rate of interest. Then $i = (1+0.02268)^4 - 1 = 9.39\%$.
Alternatively, we can use the Newton-Raphson iterations to estimate j. Letting $j_0 = \frac{2(k-n)}{k(n-1)} = \frac{2(25-20)}{25(20-1)} \approx 0.021053$ be the starting value of the interations, the first three iterations of Newton-Raphson method are

$$\begin{aligned}
j_1 =& 0.021053 \left[1 + \frac{1.021053^{20} - 1 - 25(0.021053)}{(1.021053)^{19}(1 - 0.021053 \times 19) - 1}\right] = 0.02288 \\
j_2 =& 0.02288 \left[1 + \frac{1.02288^{20} - 1 - 25(0.02288)}{(1.02288)^{19}(1 - 0.02288 \times 19) - 1}\right] = 0.02277 \\
j_3 =& 0.02277 \left[1 + \frac{1.02277^{20} - 1 - 25(0.02277)}{(1.02277)^{19}(1 - 0.02277 \times 19) - 1}\right] = 0.02285 \\
j_4 =& 0.02285 \left[1 + \frac{1.02285^{20} - 1 - 25(0.02285)}{(1.02285)^{19}(1 - 0.02285 \times 19) - 1}\right] = 0.02285
\end{aligned}$$

Thus, $j = 2.285\%$ and consequently $i = (1.02285)^4 - 1 = 0.0946 = 9.46\%$ ■

Practice Problems

Problem 22.1
Calculate the present value of an annuity due that pays 500 per month for 10 years. The annual effective interest rate is 6%.

Problem 22.2
Calculate the present value of an annuity immediate of 100 per quarter for 6 years using a nominal interest rate of 9% compounded monthly.

Problem 22.3
Calculate the accumulated value of an annuity which pays 1,000 at the beginning of each year for 10 years. Use an interest rate of $i^{(12)} = 0.08$.

Problem 22.4
A perpetuity pays 1,000 at the end of each quarter. Calculate the present value using an annual effective interest rate of 10%.

Problem 22.5
A perpetuity pays 100 at the beginning of each quarter. Calculate the present value using a force of interest of 0.06.

Problem 22.6
A perpetuity due pays 6,000 at the start of each year. Calculate the present value using $i^{(4)} = 0.06$.

Problem 22.7
Find the accumulated value at the end of four years of an investment fund in which \$100 is deposited at the beginning of each quarter for the first two years and \$200 is deposited at the beginning of each quarter for the second two years, if the fund earns 12% convertible monthly.

Problem 22.8
A 20-year annuity-due pays \$1,000 each year. If $i^{(4)} = 8\%$, find the present value of this annuity three years before the first payment.

Problem 22.9
Find the accumulated value 18 years after the first payment is made of an annuity-immediate on which there are 8 payments of \$2,000 each made at two-year intervals. The nominal interest rate is 7% convertible semiannually. Round your answer to the nearest dollar.

Problem 22.10
Find the present value of a ten-year annuity which pays \$400 at the beginning of each quarter for the first 5 years, increasing to \$600 per quarter thereafter. The annual effective rate of interest is 12%. Round to the nearest dollar.

Problem 22.11
A sum of \$100 is placed into a fund at the beginning of every other year for eight years. If the fund balance at the end of eight years is \$520, find the rate of simple interest earned by the fund.

Problem 22.12
An annuity pays 100 at the end of each quarter for 10 years. Calculate the present value of the annuity using an annual effective interest rate of 12%.

Problem 22.13
An annuity pays 100 at the end of each quarter for 10 years. Calculate the present value of the annuity using a nominal interest rate of 12% compounded monthly.

Problem 22.14
An annuity pays 100 at the end of each quarter for 10 years. Calculate the present value of the annuity using a nominal interest rate of 12% compounded six times per year.

Problem 22.15
A 30$-$year annuity pays \$1,000 every quarter. If $i^{(12)} = 6\%$, what is the present value of this annuity six months prior to the first payment?

Problem 22.16
You deposit \$2,500 into an account at the beginning of every month for 15 years. The interest rate on the account is $i^{(2)} = 8\%$. Find the accumulated value in the account nine months after the last deposit.

Problem 22.17
A perpetuity will make annual payments of \$3,000, with the first payment occurring 5 months from now. The interest rate is 12% convertible monthly. Find the present value of this annuity.

Problem 22.18
You want to accumulate \$2,000,000 over the next 30 years. You intend to do this by making deposits of X into an investment account at the end of each month, for 30 years. The account earns 12% convertible semi-annually. Find X.

23 Analysis of Annuities Payable Less Frequently than Interest is Convertible

In this section we analyze annuities where the compounding frequency is larger than the payment frequency. We start with the case of annuity−immediate.
Let k be the number of interest conversion periods in one payment period. Consider an annuity−immediate that pays 1 at the end of each payment period. Let i be the rate per conversion period and n the total number of conversion periods for the term of the annuity. We will assume that each payment period contains an integral number of interest conversion periods so that n and k are positive integers and also we assume that n is divisible by k. The total number of annuity payments made is then $\frac{n}{k}$.
Let L denote the present value of an annuity−immediate which pays 1 at the end of each k interest conversion periods for a total of n interest conversion periods. A time diagram of this sitatuation is shown in Figure 32.1.

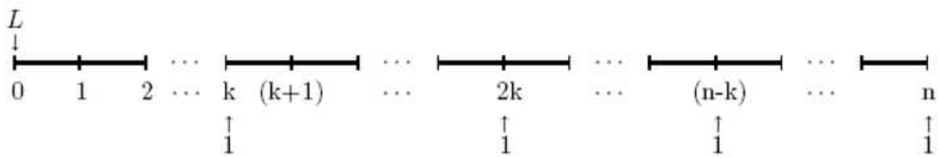


Figure 23.1

Using the equation of value at time $t=0$ we find

$$\begin{aligned}
L =&\nu^k + \nu^{2k} + \cdot + \nu^{\frac{n}{k}\cdot k}\\
=&\nu^k(1+\nu^k+\nu^{2k}+\cdot+\nu^{\left(\frac{n}{k}-1\right)k})\\
=&\nu^k \times \frac{1-(\nu^k)^{\frac{n}{k}}}{1-\nu^k} = \nu^k\frac{1-\nu^n}{1-\nu^k}\\
=&\frac{1-\nu^n}{(1+i)^k-1}\\
=&\frac{\frac{1-\nu^n}{i}}{\frac{(1+i)^k-1}{i}} = \frac{a_{\overline{n}|}}{s_{\overline{k}|}}.
\end{aligned}$$

The accumulated value of this annuity immediately after the last payment is made is

$$(1+i)^n\frac{a_{\overline{n}|}}{s_{\overline{k}|}} = \frac{s_{\overline{n}|}}{s_{\overline{k}|}}.$$

Remark 23.1
The annuity of making a payment of \$1 at the end of each k interest conversion periods for a total of $\frac{n}{k}$ payments is equivalent to the annuity that consists of a periodic payment of $\frac{1}{s_{\overline{k}|}}$ for n periods. A time diagram illustrating this annuity is given in Figure 23.2.

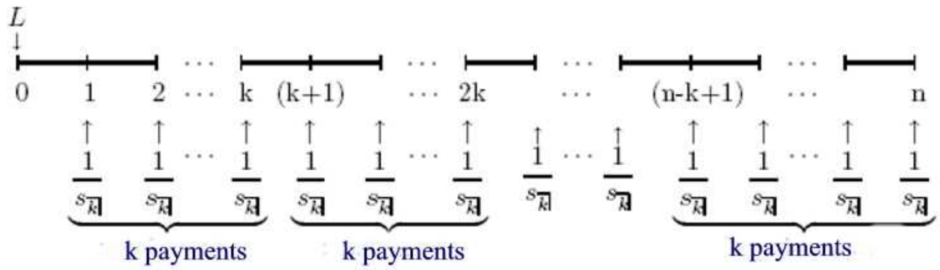


Figure 23.2

Example 23.1
Find the present value and the accumulated value of an annuity−immediate with payments of \$1,500 every 4 years from years 4 to 40. The interest earned is 8% converted annually.

Solution.
With $i = 0.08$, the present value is $1500\frac{a_{\overline{40}|}}{s_{\overline{4}|}} = 1500 \cdot \frac{11.9246}{4.5061} = \$3,969.48$ and the accumulated value is $(1.08)^{40}(3969.48) = \$86,235.05$ ■

Next, we consider the case of an annuity−due. Let $\ddot{L}$ be the present value of an annuity which pays 1 at the beginning of each k interest conversion periods for a total of n conversion interest periods. Then we have the following time diagram.

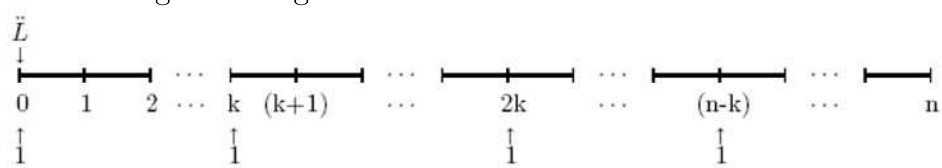


Figure 23.3

Using the equation of value at time $t = 0$ we find

$$\begin{aligned}\ddot{L} =&1+\nu^k+\nu^{2k}+\cdot+\nu^{n-k}\\ =&1+(\nu^k)+(\nu^k)^2+\cdot+(\nu^k)^{\frac{n-k}{k}}\\ =&\frac{1-(\nu^k)^{\frac{n-k}{k}+1}}{1-\nu^k}=\frac{1-\nu^n}{1-\nu^k}\\ =&\frac{\frac{1-\nu^n}{i}}{\frac{1-\nu^k}{i}}=\frac{a_{\overline{n}|}}{a_{\overline{k}|}}.\end{aligned}$$

The accumulated value of this annuity k interest conversion periods after the last payment is

$$(1+i)^n\frac{a_{\overline{n}|}}{a_{\overline{k}|}}=\frac{s_{\overline{n}|}}{a_{\overline{k}|}}.$$

Remark 23.2
It is easy to see that the annuity described above is equivalent to the annuity that consists of n payments of $\frac{1}{a_{\overline{k}|}}$ at the end of each interest conversion period.

Example 23.2
An annuity pays 1,000 at time 0, time 3, time 6, etc until 20 payments have been made. Calculate the present value and the accumulated value using an annual effective interest rate of 6%.

Solution.
With $i=0.06$, the present value is $1000\frac{a_{\overline{60}|}}{a_{\overline{3}|}}=1000\cdot\frac{16.16142771}{2.67301194}=6046.15$ and the accumulated value is $(1.06)^{60}(6046.15)=199,448.48$ ■

In the case of a perpetuity−immediate, the present value is

$$\begin{aligned}\nu^k+\nu^{2k}+\cdots=&\frac{\nu^k}{1-\nu^k}\\ =&\frac{1}{(1+i)^k-1}\\ =&\frac{1}{i\cdot s_{\overline{k}|}}=\lim_{n\to\infty}\frac{a_{\overline{n}|}}{s_{\overline{k}|}}.\end{aligned}$$

Example 23.3
The present value of a perpetuity paying 1 at the end of every 3 years is $\frac{125}{91}$. Find i.

Solution.
We have $\frac{1}{i\cdot s_{\overline{3}|}} = \frac{125}{91}$. Thus, $\frac{1}{(1+i)^3-1} = \frac{125}{91} \rightarrow (1+i)^3 = \left(\frac{6}{5}\right)^3 \rightarrow 1+i = \frac{6}{5} \rightarrow i = \frac{6}{5} - 1 = 0.2 = 20\%$ ■

In the case of a perpetuity-due, the present value is

$$\begin{aligned}1 + \nu^k + \nu^{2k} + \cdots =& \frac{1}{1-\nu^k}\\ =& \frac{1}{i}\cdot\frac{i}{1-(1+i)^{-k}}\\ =& \frac{1}{i\cdot a_{\overline{k}|}} = \lim_{n\to\infty}\frac{a_{\overline{n}|}}{a_{\overline{k}|}}.\end{aligned}$$

Example 23.4
A perpetuity paying 1 at the beginning of each year has a present value of 20. If this perpetuity is exchanged for another perpetuity paying R at the beginning of every 2 years with the same effective annual rate as the first perpetuity, find R so that the present values of the two perpetuities are equal.

Solution.
An equation of value now for the first perpetuity is $\frac{1}{d} = 20$ implying $i = \frac{1}{19}$. Also, given that $20 = \frac{R}{ia_{\overline{2}|}} = \frac{R}{1-\nu^2} = \frac{R}{(1-\nu)(1+\nu)} = \frac{R}{(0.05)(1.95)} \rightarrow R = (20)(0.0975) = 1.95$ ■

Another problem that comes under the category of annuities payable less frequently than interest is convertible is the problem of finding the value of a series of payments at a given force of interest δ. We illustrate this in the next example.

Example 23.5
Find an expression for the present value of an annuity on which payments are 100 per quarter for 5 years, just before the first payment is made, if $\delta = 0.08$.

Solution.
Let j be the rate per quarter equivalent to δ. Then $(1+j)^4 = e^{0.08}$ or $1+j = e^{0.02}$. Thus, PV $= 100\ddot{a}_{\overline{20}|j} = 100\left[\frac{1-(1+j)^{-20}}{1-(1+j)^{-1}}\right] = 100\left[\frac{1-e^{-0.40}}{1-e^{-0.02}}\right]$ ■

In some cases, the number of conversion periods in a payment period is not an integral, i.e. $k > 1$ but k is not an integer. In this case, we handle this problem using the basic principles, i.e. to find the present value or the accumulated value we write a sum of present values or accumulated values of the individual payments. An illustration of this type is shown next.

Example 23.6
Find an expression for the present value of an annuity on which payments are 1 at the beginning of each 4−month period for 12 years, assuming a rate of interest per 3−month period i.

Solution.
Let i be the effective rate of interest per 3-month period and j the effective interest rate per 4−month. Then $(1+j)^3 = (1+i)^4 \Rightarrow 1+j = (1+i)^{\frac{4}{3}}$. The present value is given by

$$\ddot{a}_{\overline{36}|j} = \frac{1-(1+j)^{-36}}{1-(1+j)^{-1}} = \frac{1-\nu^{48}}{1-\nu^{\frac{4}{3}}} \blacksquare$$

We conclude this section by pointing out that the approach discussed in this section can be generalized to finding annuities payable less frequently than interest is convertible on any date, as discussed in Section 17.

Example 23.7
Find the present value of an annuity−immediate in which there are a total of 5 payments of 1, the first to be made at the end of seven years, and the remaining four payments at three-year intervals, at an annual effective rate of 6%.

Solution.
The time diagram for this example is given in Figure 23.4.

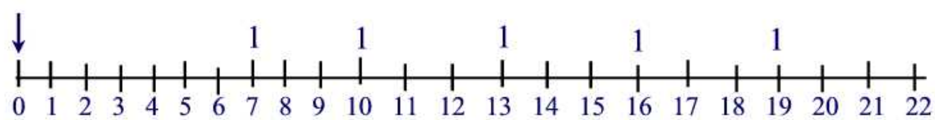


Figure 23.4

The present value is given by

$$\nu^4 \frac{a_{\overline{15}|}}{s_{\overline{3}|}} = (1.06)^{-4} \frac{1-(1.06)^{-15}}{(1.06)^3 - 1} = \$2.416 \blacksquare$$

Practice Problems

Problem 23.1
An annuity pays 1,000 at time 0, time 3, time 6, etc until 20 payments have been made. Calculate the accumulated value immediately after the last payment using an annual effective interest rate of 6%.

Problem 23.2
Give an expression of the the present value, 3 years before the first payment is made, of an annuity on which there are payments of 200 every 4 months for 12 years:
(a) expressed as an annuity-immediate;
(b) expressed as an annuity-due.
Assume monthly rate of interest.

Problem 23.3
Find an expression for the present value of an annuity-due of 600 per annum payable semiannually for 10 years, if $d^{(12)} = 0.09$.

Problem 23.4
An annuity pays 500 at the start of each year for 15 years. Calculate the accumulated value of the annuity after 15 years assuming a constant force of interest of 5%.

Problem 23.5
A perpetuity pays 100 at the end of every third year. The first payment is in three years. Calculate the present value of the perpetuity using a constant force of interest of 10%.

Problem 23.6
Find an expression for the present value of an annuity in which there are a total of r payments of 1, the first to be made at the end of seven years, and the remaining payments at three-year intervals, at an annual effective rate i, expressed as
(a) annuity-immediate
(b) annuity-due.

Problem 23.7
A 30-year annuity-immediate with 2,000 payable at the end of every 6 months and a 30-year annuity-immediate with 10,000 payable at the end of every 6 years are to be replaced by a perpetuity paying R every 3 months. You are given that $i^{(4)} = .08$. Find R.

Problem 23.8
The payments you received from a 20 year annuity−immediate paying 500 every 6 months have been left to accumulate in a fund and are now worth 40,000. If $i^{(12)} = 0.06$ is the rate earned by your fund, calculate how long it is since the last annuity payment was made.

Problem 23.9
A perpetuity of \$1,000 payable at the end of every 6 months and a perpetuity of \$10,000 payable at the end of every 6 years are to be replaced by a 30 year annuity paying R every 3 months. You are given that $i^{(4)} = 0.08$. Find R.

Problem 23.10
An investment of \$1,000 is used to make payments of \$100 at the end of each year for as along as possible with a smaller payment to be made at the time of the last regular payment. If interest is 7% convertible semiannually, find the number of payments and the amount of the total final payment.

Problem 23.11
Show that the present value at time 0 of 1 payable at times 7, 11, 15, 19, 23, and 27, where the effective rate per annum is i, is given by

$$\frac{a_{\overline{28}|} - a_{\overline{4}|}}{s_{\overline{3}|} + a_{\overline{1}|}}.$$

Problem 23.12 ‡
Fence posts set in soil last 9 years and cost \$Y each while fence posts set in concrete last 15 years and cost \$(Y + X). The posts will be needed for 35 years. Find an expression of X such that a fence builder would be indifferent between the two types of posts?

Problem 23.13
A perpetuity of \$750 payable at the end of every year and a perpetuity of \$750 payable at the end of every 20 years are to be replaced by an annuity of R payable at the end of every year for 30 years. If $i^{(2)} = 0.04$, find an expression for R.

Problem 23.14
Given that $\delta_t = \frac{2}{10+t},\ t \geq 0$, find $a_{\overline{4}|}$.

Problem 23.15
At a nominal rate of interest i, convertible semiannually, the present value of a series of payments of 1 at the end of every 2 years forever, is 5.89. Calculate i.

Problem 23.16
Gus deposits 25 into a bank account at the beginning of each 3-year period for 18 years (i.e. there is no deposit in year 18). The account credits interest at an annual effective rate of interest of i. The accumulated amount in the account after 18 years is X, which is four times as large as the accumulated amount in the account after 9 years (excluding the deposit made at time $t = 9$). Calculate X.

Problem 23.17
Find the present value of an annuity−due in which there are a total of 5 payments of 1, the first to be made at the end of seven years, and the remaining four payments at three-year intervals, at an annual effective rate of 6%.

Problem 23.18
The present value today of a 20 year annuity−immediate paying 500 every 6 months but with the first payment deferred t years has a present value of 5,805.74. If $i^{(12)} = 0.09$ is the interest rate used to calculate the present value, find t.

Problem 23.19
A 20−year annuity−due pays \$1,000 each year. If $i^{(4)} = 8\%$, find the present value of this annuity three years before the first payment.

24 Analysis of Annuities Payable More Frequently than Interest is Convertible

In a discussion parallel to the one in section 23, we discuss annuities payable more frequently than interest is convertible.
We first consider the case of an annuity-immediate. Let m denote the number of payments per one interest conversion period. Let n be the total number of conversion periods in the term of the annuity. Then the total number of payments for the term of the annuity is mn. Let i be the interest rate per conversion period. We will assume that the number of payments per conversion period is an integral number.
Payments of 1 are being made per interest conversion period with $\frac{1}{m}$ being made at the end of each mth of an interest conversion period. The present value of such an annuity will be denoted by $a_{\overline{n}|}^{(m)}$. We have the following time diagram.

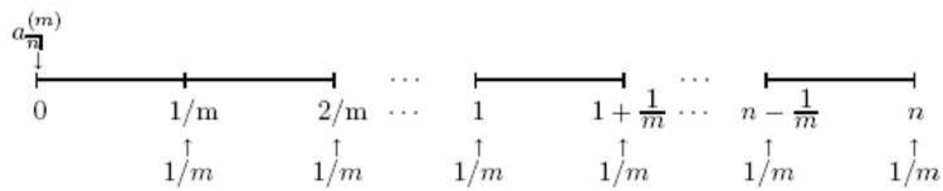


Figure 24.1

The formula for the present value is derived as follows

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\left[\nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \cdots + \nu^{n-\frac{1}{m}} + \nu^n\right] = \frac{\nu^{\frac{1}{m}}}{m}\left[\frac{1-(\nu^{\frac{1}{m}})^{mn}}{1-\nu^{\frac{1}{m}}}\right]$$
$$= \frac{1-\nu^n}{m\left[(1+i)^{\frac{1}{m}}-1\right]} = \frac{1-\nu^n}{i^{(m)}}.$$

The accumulated value of this annuity immediately after the last payment is made is given by

$$s_{\overline{n}|}^{(m)} = (1+i)^n a_{\overline{n}|}^{(m)} = \frac{(1+i)^n - 1}{i^{(m)}}.$$

Formulas of $a_{\overline{n}|}^{(m)}$ and $s_{\overline{n}|}^{(m)}$ in terms of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ are

$$a_{\overline{n}|}^{(m)} = \tfrac{i}{i^{(m)}} a_{\overline{n}|} = s_{\overline{1}|}^{(m)} a_{\overline{n}|} \qquad \text{and} \qquad s_{\overline{n}|}^{(m)} = \tfrac{i}{i^{(m)}} s_{\overline{n}|} = s_{\overline{1}|}^{(m)} s_{\overline{n}|}.$$

One consideration which is important in practice involves the proper coefficients for annuity payable $m$thly. Each payment made is of amount $\frac{1}{m}$, while the coefficient of the symbol $a_{\overline{n}|}^{(m)}$ is 1. In general, the proper coefficient is the total amount paid during one interest conversion period, and not the amount of each actual payment. The total amount paid during one interest conversion period is called **periodic rent** of the annuity.

Example 24.1
A loan of $3,000 is to be repaid with quarterly installments at the end of each quarter for five years. If the rate of interest charged is 10% converted semiannually, find the amount of each quarterly payment.

Solution.
Let R be the amount of quarterly payment. In one interest conversion period there are two payments. Then $2Ra_{\overline{10}|0.05}^{(2)} = 3000 \rightarrow R = \frac{1500}{\frac{1-(1.05)^{-10}}{2[(1.05)^{0.5}-1]}} = 191.89$ ■

Example 24.2
What is the accumulated value following the last payment of $100 deposited at the end of each month for 30 years into an account that earns an effective annual rate of 10%?

Solution.
Each conversion period contains 12 payment periods. The term of the loan is 30 interest conversion periods. Thus, $12 \cdot 100 \cdot \frac{(1.1)^{30}-1}{12[(1.1)^{\frac{1}{12}}-1]} = 206,284.33$ ■

Example 24.3
The accumulated amount of an annuity-immediate of $R per year payable quarterly for seven years is $3317.25. Find R if $i^{(1)} = 0.05$.

Solution.
Each R is paid quarterly at $\frac{R}{4}$. The equation of value at time $t = 7$ years is

$$Rs_{\overline{7}|}^{(4)} = 3317.25.$$

But

$$s_{\overline{7}|}^{(4)} = \frac{i}{i^{(4)}} s_{\overline{7}|}.$$

On the other hand,

$$\left(1 + \frac{i^{(4)}}{4}\right)^4 = 1.05 \Rightarrow i^{(4)} = 4[(1.05)^{\frac{1}{4}} - 1] = 0.0491.$$

Hence, we obtain $3317.25 = R\frac{0.05}{0.0491}(8.1420)$. Solving for R; we find $R = \$400$ ■

In the case of annuity-due, the amount $\frac{1}{m}$ is payable at the beginning of the mth period of an interest conversion period for a total of n interest conversion period. The present value is denoted by $\ddot{a}_{\overline{n}|}^{(m)}$. A time diagram of this case is shown in Figure 24.2.

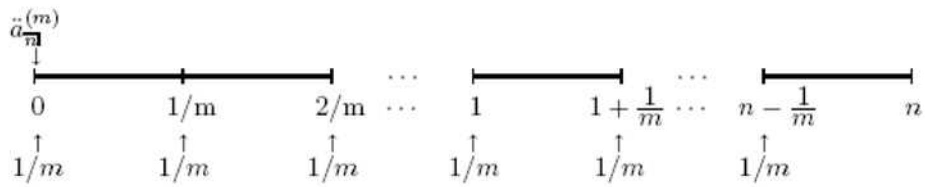


Figure 24.2

The present value is determined as follows:

$$\begin{aligned}\ddot{a}_{\overline{n}|}^{(m)} =& \frac{1}{m}[1+\nu^{\frac{1}{m}}+\nu^{\frac{2}{m}}+\cdots+\nu^{n-\frac{1}{m}}]\\ =&\frac{1}{m}\left[\frac{1-(\nu^{\frac{1}{m}})^{mn}}{1-\nu^{\frac{1}{m}}}\right]\\ =&\frac{1-\nu^n}{m[1-(1+i)^{-\frac{1}{m}}]}=\frac{1-\nu^n}{d^{(m)}}.\end{aligned}$$

The accumulated value one mth of an interest conversion period after the last payment is made is given by

$$\ddot{s}_{\overline{n}|}^{(m)} = (1+i)^n\ddot{a}_{\overline{n}|}^{(m)} = \frac{(1+i)^n-1}{d^{(m)}}.$$

It follows from the formulas of $a_{\overline{n}|}, s_{\overline{n}|}, \ddot{a}_{\overline{n}|}^{(m)}$, and $\ddot{s}_{\overline{n}|}^{(m)}$ that

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{i}{d^{(m)}}a_{\overline{n}|} = \ddot{s}_{\overline{1}|}^{(m)}a_{\overline{n}|}$$

and

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{i}{d^{(m)}}s_{\overline{n}|} = \ddot{s}_{\overline{1}|}^{(m)}s_{\overline{n}|}.$$

Example 24.4
What is the present value of the payment of \$150 per quarter for 10 years if the effective annual interest rate is 6% and the first payment is due today.

Solution.
The answer is $600\cdot \ddot{a}^{(4)}_{\overline{10}|0.06} = 600\cdot a_{\overline{10}|0.06}\cdot \frac{i}{d^{(4)}} = 600\cdot 7.3601\cdot 1.037227 = \$4,580.46$ ■

Example 24.5
Express $\ddot{a}^{(12)}_{\overline{n}|}$ in terms of $a^{(2)}_{\overline{n}|}$ with an adjustment factor.

Solution.
We have

$$\ddot{a}^{(12)}_{\overline{n}|} = \frac{i}{d^{(12)}}a_{\overline{n}|} \text{ and } a^{(2)}_{\overline{n}|} = \frac{i}{i^{(2)}}a_{\overline{n}|}.$$

Thus,

$$\ddot{a}^{(12)}_{\overline{n}|} = \frac{i^{(2)}}{d^{(12)}}a^{(2)}_{\overline{n}|} \blacksquare$$

The following identities involving $a_{\overline{n}|}, s_{\overline{n}|}, \ddot{a}^{(m)}_{\overline{n}|}$, and $\ddot{s}^{(m)}_{\overline{n}|}$ are analogous to the identity involving $a_{\overline{n}|}, s_{\overline{n}|}, \ddot{a}_{\overline{n}|}$, and $\ddot{s}_{\overline{n}|}$ discussed in earlier sections.

Theorem 24.1
(a) $\frac{1}{a^{(m)}_{\overline{n}|}} = \frac{1}{s^{(m)}_{\overline{n}|}} + i^{(m)}$

(b) $\frac{1}{\ddot{a}^{(m)}_{\overline{n}|}} = \frac{1}{\ddot{s}^{(m)}_{\overline{n}|}} + d^{(m)}$

(c) $\ddot{a}^{(m)}_{\overline{n}|} = (1+i)^{\frac{1}{m}}a^{(m)}_{\overline{n}|} = \left[\frac{i}{i^{(m)}} + \frac{i}{m}\right]a_{\overline{n}|}$

(d) $\ddot{s}^{(m)}_{\overline{n}|} = (1+i)^{\frac{1}{m}}s^{(m)}_{\overline{n}|} = \left[\frac{i}{i^{(m)}} + \frac{i}{m}\right]s_{\overline{n}|}$

(e) $\ddot{a}^{(m)}_{\overline{n}|} = \frac{1}{m} + a^{(m)}_{\overline{n-\frac{1}{m}}|}$

(f) $\ddot{s}^{(m)}_{\overline{n}|} = s^{(m)}_{\overline{n+\frac{1}{m}}|} - \frac{1}{m}$

Proof.
See Problem 24.10 ■

Next, consider an infinite payment of $\frac{1}{m}$ at the end of mth of an interest conversion period. Let $a^{(m)}_{\overline{\infty}|}$ denote the present value. A time diagram describing this case is shown in Figure 24.3.

The present value is determined as follows.

$$\begin{aligned}a_{\overline{\infty}|}^{(m)} =&\frac{1}{m}[\nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \nu^{\frac{3}{m}} + \cdots] = \frac{1}{m}\sum_{p=1}^{\infty}\nu^{\frac{p}{m}}\\ =&\frac{1}{m}\frac{\nu^{\frac{1}{m}}}{1-\nu^{\frac{1}{m}}} = \frac{1}{m(1+i)^{\frac{1}{m}}(1-\nu^{\frac{1}{m}})}\\ =&\frac{1}{m((1+i)^{\frac{1}{m}}-1)} = \frac{1}{i^{(m)}}\end{aligned}$$

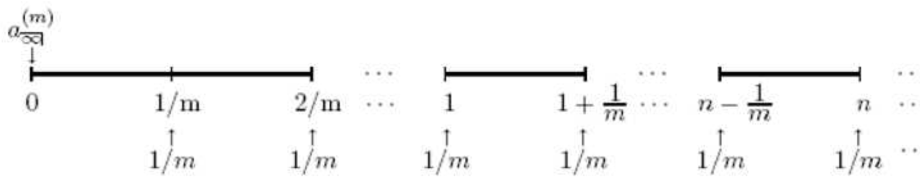


Figure 24.3

Example 24.6
An annuity pays 100 at the end of each month forever. Calculate the present value of the annuity using an annual effective interest rate of 8%.

Solution.
The present value is $12 \cdot 100a_{\overline{\infty}|}^{(12)} = \frac{1200}{12[(1.08)^{\frac{1}{12}}-1]} = 15,542.36$ ■

Now, consider an infinite payment of $\frac{1}{m}$ at the beginning of mth of an interest conversion period. Let the present value be denoted by $\ddot{a}_{\overline{\infty}|}^{(m)}$. Figure 24.4 describes this situation.

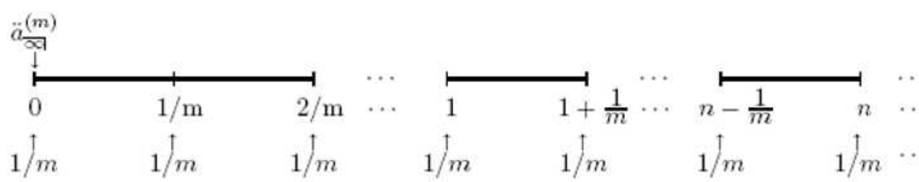


Figure 24.4

The present value is determined as follows.

$$\begin{aligned}\ddot{a}_{\overline{\infty}|}^{(m)} =&\frac{1}{m}[1 + \nu^{\frac{1}{m}} + \nu^{\frac{2}{m}} + \nu^{\frac{3}{m}} + \cdots] = \frac{1}{m}\sum_{p=0}^{\infty}\nu^{\frac{p}{m}}\\ =&\frac{1}{m[1-\nu^{\frac{1}{m}}]} = \frac{1}{d^{(m)}},\ \ \nu^{\frac{1}{m}} < 1\end{aligned}$$

Example 24.7
At what annual effective rate of interest is the present value of a series of payments of \$1 for every six months forever, with the first payment made now, equal to \$10?

Solution.
We have $10 = 2\ddot{a}^{(2)}_{\overline{\infty}|} = \frac{2}{d^{(2)}} \rightarrow d^{(2)} = 0.2 \rightarrow i = (1-0.1)^{-2} - 1 = 23.46\%$ ■

In some cases, each conversion period does not contain an integral number of payments, i.e. $m > 1$ but m is not an integer. In this case, we handle this problem using the basic principles, i.e. to find the present value or the accumulated value we write a sum of present values or accumulated values of the individual payments. An illustration of this type is shown next.

Example 24.8
Find an expression for the present value of an annuity which pays 1 at the beginning of each 3-month period for 12 years, assuming a rate of interest per 4-month period.

Solution.
Working in years the annual payments are $4 \times 1 = \$4$. Thus, the present value is

$$4\ddot{a}^{(4)}_{\overline{12}|} = 4 \cdot \frac{1-(1+i)^{-12}}{d^{(4)}}.$$

But

$$1+i = \left(1 - \frac{d^{(4)}}{4}\right)^{-4} \Rightarrow d^{(4)} = 4[1-(1+i)^{-\frac{1}{4}}].$$

Thus

$$\begin{aligned}4\ddot{a}^{(4)}_{\overline{10}|} =& 4\frac{1-(1+i)^{-12}}{4[1-(1+i)^{-\frac{1}{4}}]}\\ =&\frac{1-\nu^{12}}{1-\nu^{\frac{1}{4}}}\\ =&\frac{1-\nu_j^{48}}{1-\nu_j}\\ =&\frac{1-\nu_k^{36}}{1-\nu_k^{\frac{3}{4}}}\end{aligned}$$

where j is the effective interest rate per 3-month and k is the effective interest rate per 4-month ■

Finally, it is possible to generalize the approach to finding annuity values on any date, as discussed in Section 17, to annuities payable more frequently than interest is convertible.

Example 24.9
Payments of $400 at the end of each month are made over a ten-year period. Find expressions for
(a) the present value of these payments two years prior to the first payment;
(b) the accumulated value three years after the final payment.
Use symbols based on an effective rate of interest i.

Solution.
(a) The present value is given by $12 \cdot 400\nu a_{\overline{10}|}^{(12)}$.
(b) The accumulated value is given by $12 \cdot 400 s_{\overline{10}|}^{(12)}(1+i)^3$ ■

Practice Problems

Problem 24.1
The present value of an annuity is denoted by $120a_{\overline{n}|}^{(12)}$. What is the amount of the monthly payment of this annuity?

Problem 24.2
The present value of an annuity that pays 100 at the end of each year for n years using an annual effective interest rate of 10.25% is 1,000. Calculate the present value of an annuity that pays 100 at the end of every six months for n years using the same interest rate. (Note: You should work this problem without finding n.)

Problem 24.3
Suppose you deposit \$200 per month, at the end of each month, in a savings account paying the effective annual rate of interest 6%. How much will be in the account after ten years? (i.e. just after your 120th deposit?)

Problem 24.4
A loan of \$150,000 is repaid by equal installments at the end of each month, for 25 years. Given that the nominal annual rate of interest, convertible semiannually, is 8%, compute the amount of monthly installment.

Problem 24.5
Suppose you deposit \$300 at the end of every three months for 10 years into a bank account paying the annual effective rate of interest 4%. What is the accumulated amount after 10 years?

Problem 24.6
An annuity pays 100 at the end of each month for 20 years. Using a nominal rate of interest of 4% compounded quarterly, calculate the current value of the annuity at the end of the 3rd year.

Problem 24.7
Which of the following are true?
(i) $a_{\overline{n}|} = \frac{i^{(m)}}{i} a_{\overline{n}|}^{(m)}$
(ii) $s_{\overline{n}|}^{(m)} = (1+i)^n a_{\overline{n}|}^{(m)}$
(iii) $a_{\overline{\infty}|}^{(\infty)} = \ddot{a}_{\overline{\infty}|}^{(\infty)}$

Problem 24.8
If $3a_{\overline{n}|}^{(2)} = 2a_{\overline{2n}|}^{(2)} = 45s_{\overline{1}|}^{(2)}$, find i.

Problem 24.9
A sum of 10,000 is used to buy a deferred perpetuity-due paying \$500 every six months forever. Find an expression for the deferred period expressed as a function of d, the annual discount rate.

Problem 24.10
Prove the following identities:
(a) $\frac{1}{a_{\overline{n}|}^{(m)}} = \frac{1}{s_{\overline{n}|}^{(m)}} + i^{(m)}$

(b) $\frac{1}{\ddot{a}_{\overline{n}|}^{(m)}} = \frac{1}{\ddot{s}_{\overline{n}|}^{(m)}} + d^{(m)}$

(c) $\ddot{a}_{\overline{n}|}^{(m)} = (1+i)^{\frac{1}{m}} a_{\overline{n}|}^{(m)} = \left[\frac{i}{i^{(m)}} + \frac{i}{m}\right] a_{\overline{n}|}$

(d) $\ddot{s}_{\overline{n}|}^{(m)} = (1+i)^{\frac{1}{m}} s_{\overline{n}|}^{(m)} = \left[\frac{i}{i^{(m)}} + \frac{i}{m}\right] s_{\overline{n}|}$

(e) $\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m} + a_{\overline{n-\frac{1}{m}}|}^{(m)}$

(f) $\ddot{s}_{\overline{n}|}^{(m)} = s_{\overline{n+\frac{1}{m}}|}^{(m)} - \frac{1}{m}$

Problem 24.11
Find the present value of a ten-year annuity which pays \$400 at the beginning of each quarter for the first 5 years, increasing to \$600 per quarter thereafter. The annual effective rate of interest is 12%. Round to the nearest dollar. Use the approach developed in this section.

Problem 24.12
A family wishes to provide an annuity of \$100 at the end of each month to their daughter now entering college. The annuity will be paid for only nine months each year for four years. Prove that the present value one month before the first payment is

$$1200\ddot{a}_{\overline{4}|}a_{\overline{9/12}|}^{(12)}.$$

Problem 24.13
The nominal rate of interest, convertible semiannually, is 6%. An annuity−immediate pays \$50 each month for five years. Find the accumulated value of this annuity at the time of the last payment.

Problem 24.14
You deposit \$2,500 into an account at the beginning of every month for 15 years. The interest rate on the account is $i^{(2)} = 8\%$. Find the accumulated value in the account nine months after the last deposit.

Problem 24.15
You want to accumulate \$1,000,000. You intend to do this by making deposits of \$5,000 into an investment account at the end of each month, until your account balance equals \$1,000,000. The account earns 8% convertible quarterly. Determine the number of monthly deposits you will need to make to achieve your goal.

25 Continuous Annuities

In this section we consider annuities with a finite term and an infinte frequency of payments . Formulas corresponding to such annuities are useful as approximations corresponding to annuities payable with great frequency such as daily.
Consider an annuity in which a very small payment dt is made at time t and these small payments are payable continuously for n interest conversion periods. Let i denote the periodic interest rate. Then the total amount paid during each period is

$$\int_{k-1}^{k} dt = [t]_{k-1}^{k} = \$1.$$

Let $\overline{a}_{\overline{n}|}$ denote the present value of an annuity payable continuously for n interest conversion periods so that 1 is the total amount paid during each interest conversion period. Then the present value can be found as follows:

$$\begin{aligned}\overline{a}_{\overline{n}|} =& \int_0^n \nu^t dt = \left.\frac{\nu^t}{\ln \nu}\right|_0^n \\ =&\frac{\nu^n - 1}{\ln \nu} = \frac{\nu^n - 1}{-\ln(1+i)} \\ =&\frac{1-\nu^n}{\delta}.\end{aligned}$$

It is easy to see the following

$$\overline{a}_{\overline{n}|} = \lim_{m\to\infty} a_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \frac{1-\nu^n}{i^{(m)}} = \frac{1-\nu^n}{\delta}$$

since

$$\lim_{m\to\infty} i^{(m)} = \delta.$$

Similarly,

$$\overline{a}_{\overline{n}|} = \lim_{m\to\infty} \ddot{a}_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \frac{1-\nu^n}{d^{(m)}} = \frac{1-\nu^n}{\delta}$$

since

$$\lim_{m\to\infty} d^{(m)} = \delta.$$

Moreover,

$$\overline{a}_{\overline{n}|} = \frac{i}{\delta} a_{\overline{n}|} = \frac{d}{\delta} \ddot{a}_{\overline{n}|} = \frac{1-e^{-n\delta}}{\delta}.$$

Example 25.1
Starting four years from today, you will receive payment at the rate of \$1,000 per annum, payable continuously, with the payment terminating twelve years from today. Find the present value of this continuous annuity if $\delta = 5\%$.

Solution.
The present value is $PV = 1000v^4 \cdot \overline{a}_{\overline{8}|} = 1000e^{-0.20} \cdot \frac{1-e^{-0.40}}{0.05} = \$5,398.38$ ■

The next example exhibit the situation when the constant force above is replaced by a variable force of interest.

Example 25.2
Find an expression for $\overline{a}_{\overline{n}|}$ if $\delta_t = \frac{1}{1+t}$.

Solution.
The present value of the payment dt at the exact time t with variable force of interest is $e^{-\int_0^t \delta_r dr}dt = e^{-\ln(1+t)}dt = (1+t)^{-1}dt$. Hence,

$$\overline{a}_{\overline{n}|} = \int_0^n \frac{dt}{1+t} = \ln(1+t)|_0^n = \ln(1+n) \text{ ■}$$

Next, let $\overline{s}_{\overline{n}|}$ denote the accumulated value at the end of the term of an annuity payable continuously for n interest conversion periods so that 1 is the total amound paid during each interest conversion period. Then

$$\begin{aligned}\overline{s}_{\overline{n}|} =&(1+i)^n\overline{a}_{\overline{n}|}\\ =&\int_0^n (1+i)^{n-t}dt\\ =&-\left.\frac{(1+i)^{n-t}}{\ln(1+i)}\right|_0^n\\ =&\frac{(1+i)^n-1}{\delta}.\end{aligned}$$

It is easy to see that

$$\overline{s}_{\overline{n}|} = \lim_{m\to\infty} s_{\overline{n}|}^{(m)} = \lim_{m\to\infty} \ddot{s}_{\overline{n}|}^{(m)} = \frac{e^{n\delta}-1}{\delta} = \frac{i}{\delta}s_{\overline{n}|} = \frac{d}{\delta}\ddot{s}_{\overline{n}|}.$$

Example 25.3
Find the force of interest at which the accumulated value of a continuous payment of 1 every year for 8 years will be equal to four times the accumulated value of a continuous payment of 1 every year for four years.

Solution.
We have

$$\begin{aligned}\overline{s}_{\overline{8}|} =& 4\overline{s}_{\overline{4}|}\\ \frac{e^{8\delta}-1}{\delta} =& 4\cdot\frac{e^{4\delta}-1}{\delta}\\ e^{8\delta}-4e^{4\delta}+3 =& 0\\ (e^{4\delta}-3)(e^{4\delta}-1) =& 0\end{aligned}$$

If $e^{4\delta}=3$ then $\delta=\frac{\ln 3}{4}\approx 0.275=27.5\%$. If $e^{4\delta}=1$ then $\delta=0$, an extraneous solution ■

Example 25.4
The annual effective rate is $i=5\%$. Find the accumulated value after six years of an annuity that offers payments of \$1,000 per annum, convertible continuously for five years. After five years, payments terminate, but the balance still earns interest during the sixth year.

Solution.
The accumulated value is $AV=1000\overline{s}_{\overline{5}|}(1.05)$. But $\delta=\ln(1+i)=\ln 1.05=0.0487902$. Thus, $AV=1000\cdot\frac{(1.05)^5-1}{0.0487902}\cdot(1.05)=\$5,945.78$ ■

The present value of a perpetuity payable continuously with total of 1 per period is given by

$$\overline{a}_{\overline{\infty}|}=\lim_{n\to\infty}\overline{a}_{\overline{n}|}=\frac{1}{\delta}.$$

Example 25.5
A perpetuity paid continuously at a rate of 100 per year has a present value of 800. Calculate the annual effective interest rate used to calculate the present value.

Solution.
The equation of value at time $t=0$ is

$$800=\frac{100}{\delta}=\frac{100}{\ln(1+i)}.$$

Thus,

$$i=e^{\frac{1}{8}}-1=13.3\%\ ■$$

Practice Problems

Problem 25.1
Calculate the present value of a continuous annuity of 1,000 per annum for 8 years at:
(a) An annual effective interest rate of 4%;
(b) A constant force of interest of 4%.

Problem 25.2
Find the force of interest at which $\overline{s}_{\overline{20}|} = 3\overline{s}_{\overline{10}|}$.

Problem 25.3
If $\overline{a}_{\overline{n}|} = 4$ and $\overline{s}_{\overline{n}|} = 12$, find δ.

Problem 25.4
There is \$40,000 in a fund which is accumulating at 4% per annum convertible continuously. If money is withdrawn continuously at the rate of \$2,400 per annum, how long will the fund last?

Problem 25.5
Annuity A offers to pay you \$100 per annum, convertible continuously, for the next five years. Annuity B offers you \$ X at the end of each year for ten years. The annual effective interest rate i is 8%. Find X such that you are indifferent between the two annuities.

Problem 25.6
Given $\delta = 0.1$. Evaluate $\frac{a^{(12)}_{\overline{10}|}}{\overline{a}_{\overline{10}|}}$.

Problem 25.7
You are given $\frac{d}{dt}\overline{s}_{\overline{t}|} = (1.02)^{2t}$. Calculate δ.

Problem 25.8
Given $\overline{a}_{\overline{n}|} = n - 4$ and $\delta = 10\%$, find $\int_0^n \overline{a}_{\overline{t}|} dt$.

Problem 25.9
Payments will be made to you at a continuous rate of \$100 per year for the next five years. The effective rate of interest over this period is $i = 0.06$. Calculate the present value of this payment stream.

Problem 25.10
Show that $\frac{d}{dt}a_{\overline{t}|} = \frac{\nu^t}{\overline{s}_{\overline{1}|}}$.

Problem 25.11
Show that $a_{\overline{n}|} < a_{\overline{n}|}^{(m)} < \overline{a}_{\overline{n}|} < \ddot{a}_{\overline{n}|}^{(m)} < \ddot{a}_{\overline{n}|}$. Hint: See Example 10.15.

Problem 25.12
Find an expression for $t,\ 0 < t < 1$, such that 1 paid at time t is equivalent to 1 paid continuously between 0 and 1.

Problem 25.13
A bank makes payments continuously at a rate of \$400 a year. The payments are made between 5 and 7 years. Find the current value of these payments at time 2 years using an annual rate of discount of 4%.

Problem 25.14
A company makes payments continuously at a rate of \$200 per year. The payments are made between 2 and 7 years. Find the accumulated value of these payments at time 10 years using an annual rate of interest of 6.5%.

Problem 25.15
Lauren is being paid a continuous perpetuity payable at a rate of 1000 per year. Calculate the present value of the perpetuity assuming $d^{(12)} = 0.12$.

Problem 25.16
If $i = 0.04$, calculate the accumulated value of a continuous annuity payable at a rate of 100 per year for 10 years.

Problem 25.17
If $\delta = 0.06$, calculate the present value of a continuous annuity of 1 payable for 20 years.

Problem 25.18
You are given $\int_0^n \overline{a}_{\overline{t}|}dt = 100$. Calculate $\overline{a}_{\overline{n}|}$.

Problem 25.19
An $n-$year continuous annuity that pays at a rate of \$748 per year has a present value of \$10,000 when using an interest rate of 6%, compounded continuously. Determine n.

Problem 25.20
Which of the following are true?

(I) $\left(\overline{a}_{\overline{n}|} - \frac{d}{\delta}\right)(1+i) = \overline{a}_{\overline{n-1}|}$.

(II) The present value of a 10 year annuity immediate paying 10 per month for the first eight months of each year is $120a_{\overline{10}|}a^{(12)}_{\overline{8/12}|}$.

(III) The present value of a perpetuity paying one at the end of each year, except paying nothing every fourth year, is $\frac{s_{\overline{3}|}}{is_{\overline{4}|}}$.

Problem 25.21
Payments of \$3650 per year are made continuously over a five-year period. Find the present value of this continuous annuity two years prior to the first payment. Given the nominal rate $i^{(365)} = 8\%$.

26 Varying Annuity-Immediate

Thus far in this book, all annuities that we have considered had level series of payments, that is, payments are all equal in values. In this and the next three sections we consider annuities with a varying series of payments. In this section, we assume that the payment period and interest conversion period coincide. Annuities with varying payments will be called **varying annuities**.
Any type of annuities can be evaluated by taking the present value or the accumulated value of each payment seperately and adding the results. There are, however, several types of varying annuities for which relatively simple compact expressions are possible. The only general types we will study vary in either arithmetic progression or geometric progression.

Payments Varying in an Arithmetic Progression
First, let us assume that payments vary in arithmetic progression. That is, the first payment is P and then the payments increase by Q thereafter, continuing for n years as shown in the time diagram of Figure 26.1.

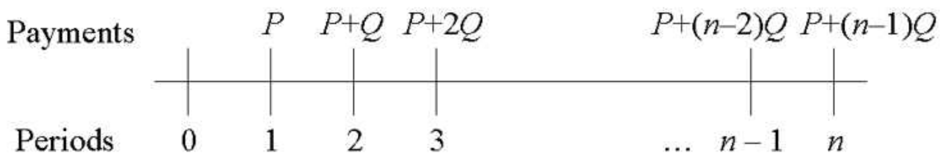


Figure 26.1

If PV is the present value for this annuity-immediate, then

$$PV = P\nu + (P+Q)\nu^2 + (P+2Q)\nu^3 + \cdots + [P+(n-1)Q]\nu^n. \tag{26.1}$$

Multiplying Equation (26.1) by $(1+i)$, we obtain

$$(1+i)PV = P + (P+Q)\nu + (P+2Q)\nu^2 + \cdots + [P+(n-1)Q]\nu^{n-1}. \tag{26.2}$$

Subtracting Equation(26.1) from Equation (26.2) we obtain

$$iPV = P(1-\nu^n) + (\nu + \nu^2 + \cdots + \nu^n)Q - n\nu^n Q$$

or

$$PV = Pa_{\overline{n|}} + Q\frac{[a_{\overline{n|}} - n\nu^n]}{i}. \tag{26.3}$$

The accumulated value of these payments at time n is

$$AV = (1+i)^n PV = Ps_{\overline{n|}} + Q\frac{[s_{\overline{n|}} - n]}{i}. \tag{26.4}$$

Two special cases of the above varying annuity often occur in practice. The first of these is the **increasing annuity** where $P = Q = 1$ as shown in Figure 26.2.

Figure 26.2

The present value of such an annuity is

$$(Ia)_{\overline{n}|} = a_{\overline{n}|} + \frac{a_{\overline{n}|} - n\nu^n}{i} = \frac{(1+i)a_{\overline{n}|} - n\nu^n}{i} = \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{i}. \tag{26.5}$$

The accumulated value at time n is given by

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i}.$$

Example 26.1
The following payments are to be received: \$500 at the end of the first year, \$520 at the end of the second year, \$540 at the end of the third year and so on, until the final payment is \$800. Using an annual effective interest rate of 2%
(a) determine the present value of these payments at time 0;
(b) determine the accumulated value of these payments at the time of the last payment.

Solution.
In n years the payment is $500 + 20(n-1)$. So the total number of payments is 16. The given payments can be regarded as the sum of a level annuity immediate of \$480 and an increasing annuity-immediate $\$20, \$40, \cdots, \$320$.
(a) The present value at time $t = 0$ is

$$480a_{\overline{16}|} + 20(Ia)_{\overline{16}|} = 480(13.5777) + 20(109.7065) = \$8,711.43.$$

(b) The accumulated value at time $t = 16$ is

$$480s_{\overline{16}|} + 20(Is)_{\overline{16}|} = 480(18.6393) + 20(150.6035) = \$11,958.93 \blacksquare$$

Example 26.2
Show that $(Ia)_{\overline{n}|} = \sum_{t=0}^{n-1} \nu^t a_{\overline{n-t}|}$.

Solution.
We have

$$\begin{aligned}(Ia)_{\overline{n}|} =& \sum_{t=0}^{n-1} \nu^t a_{\overline{n-t}|} = \sum_{t=0}^{n-1} \nu^t \frac{1-\nu^{n-t}}{i} \\ =& \frac{1}{i}\sum_{t=0}^{n-1} \nu^t - \frac{\nu^n}{i}\sum_{t=0}^{n-1} 1 \\ =& \frac{\ddot{a}_{\overline{n}|}}{i} - \frac{n\nu^n}{i} = \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{i}. \blacksquare\end{aligned}$$

The second special case is the **decreasing annuity-immediate** where $P = n$ and $Q = -1$ as shown in Figure 26.3.

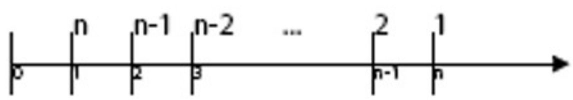


Figure 26.3

In this case, the present value one year before the first payment (i.e., at time $t = 0$) is given by

$$(Da)_{\overline{n}|} = na_{\overline{n}|} - \frac{a_{\overline{n}|} - n\nu^n}{i} = \frac{n - n\nu^n - a_{\overline{n}|} + n\nu^n}{i} = \frac{n - a_{\overline{n}|}}{i}. \tag{26.6}$$

The accumulated value at time n is given by

$$(Ds)_{\overline{n}|} = (1+i)^n (Da)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i} = (n+1)a_{\overline{n}|} - (Ia)_{\overline{n}|}.$$

Example 26.3
John receives \$400 at the end of the first year, \$350 at the end of the second year, \$300 at the end of the third year and so on, until the final payment of \$50. Using an annual effective rate of 3.5%, calculate the present value of these payments at time 0.

Solution.
In year n the payment is $400-50(n-1)$. Since the final payment is 50 we must have $400-50(n-1) = 50$. Solving for n we find $n = 8$. Thus, the present value is

$$50(Da)_{\overline{8}|} = 50 \cdot \frac{8 - a_{\overline{8}|}}{0.035} = \$1,608.63 \blacksquare$$

Example 26.4
Calculate the accumulated value in Example 26.3.

Solution.
The answer is $50(Ds)_{\overline{8}|} = 50 \cdot \frac{8(1.035)^8 - s_{\overline{8}|}}{0.035} = \$2,118.27$ ■

Example 26.5
Show that $(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} = (n+1)a_{\overline{n}|}$.

Solution.
We have

$$\begin{aligned}(Ia)_{\overline{n}|} + (Da)_{\overline{n}|} =& a_{\overline{n}|} + \frac{a_{\overline{n}|} - n\nu^n}{i} + \frac{n - a_{\overline{n}|}}{i} \\ =& a_{\overline{n}|} + na_{\overline{n}|} = (n+1)a_{\overline{n}|} \blacksquare\end{aligned}$$

Besides varying annuities immediate, it is also possible to have varying perpetuity−immediate. Consider a perpetuity−immediate with payments that form an arithmetic progression (and of course $P > 0$ and $Q > 0$). The present value for such a perpetuity with the first payment at the end of the first period is

$$\begin{aligned}PV =& \lim_{n\to\infty}\left[Pa_{\overline{n}|} + Q\frac{[a_{\overline{n}|} - n\nu^n]}{i}\right] == P\lim_{n\to\infty} a_{\overline{n}|} + Q\lim_{n\to\infty}\frac{[a_{\overline{n}|} - n\nu^n]}{i} \\ =& P\lim_{n\to\infty} a_{\overline{n}|} + Q\frac{[\lim_{n\to\infty} a_{\overline{n}|} - \lim_{n\to\infty} n\nu^n]}{i} \\ =& Pa_{\overline{\infty}|} + \frac{Qa_{\overline{\infty}|}}{i} = \frac{P}{i} + \frac{Q}{i^2}\end{aligned}$$

since $a_{\overline{\infty}|} = \frac{1}{i}$ and $\lim_{n\to\infty} n\nu^n = 0$ (by L'Hopital's rule).
For the special case $P = Q = 1$ we find

$$(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2}.$$

Example 26.6
Find the present value of a perpetuity-immediate whose successive payments are 1, 2, 3, 4,$\cdots$ at an effective rate of 6%.

Solution.
We have $(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2} = \frac{1}{0.06} + \frac{1}{0.06^2} = \$294.44$ ■

Payments Varying in a Geometric Progression

Next, we consider payments varying in a geometric progression. Consider an annuity-immediate with a term of n periods where the interest rate is i per period, and where the first payment is 1 and successive payments increase in geometric progression with common ratio $1+k$. The present value of this annuity is

$$\begin{aligned}\nu + \nu^2(1+k) + \nu^3(1+k)^2 + \cdots + \nu^n(1+k)^{n-1} =&\nu \cdot \frac{1-[(1+k)\nu]^n}{1-(1+k)\nu}\\ =&\frac{1}{1+i}\cdot\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\left(\frac{1+k}{1+i}\right)}\\ =&\frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}\end{aligned}$$

provided that $k \neq i$. If $k = i$ then the original sum consists of a sum of n terms of ν which equals to $n\nu$.

Example 26.7

The first of 30 payments of an annuity occurs in exactly one year and is equal to \$500. The payments increase so that each payment is 5% greater than the preceding payment. Find the present value of this annuity with an annual effective rate of interest of 8%.

Solution.

The present value is given by

$$PV = 500 \cdot \frac{1-\left(\frac{1.05}{1.08}\right)^{30}}{0.08-0.05} = \$9,508.28 \blacksquare$$

For an annuity-immediate with a term of n periods where the interest rate is i per period, and where the first payment is 1 and successive payments decrease in geometric progression with common ratio $1-k$. The present value of this annuity is

$$\begin{aligned}\nu + \nu^2(1-k) + \nu^3(1-k)^2 + \cdots + \nu^n(1-k)^{n-1} =&\nu \cdot \frac{1-[(1-k)\nu]^n}{1-(1-k)\nu}\\ =&\frac{1}{1+i}\cdot\frac{1-\left(\frac{1-k}{1+i}\right)^n}{1-\left(\frac{1-k}{1+i}\right)}\\ =&\frac{1-\left(\frac{1-k}{1+i}\right)^n}{i+k}\end{aligned}$$

provided that $k \neq i$. If $k = i$ then the original sum becomes

$$\nu + \nu^2(1-i) + \nu^3(1-i)^2 + \cdots + \nu^n(1-i)^{n-1} = \frac{1}{2i}\left[1 - \left(\frac{1-i}{1+i}\right)^n\right]$$

Finally, we consider a perpetuity with payments that form a geometric progression where $0 < 1+k < 1+i$. The present value for such a perpetuity with the first payment at the end of the first period is

$$\nu + \nu^2(1+k) + \nu^3(1+k)^2 + \cdots = \frac{\nu}{1-(1+k)\nu} = \frac{1}{i-k}.$$

Observe that the value for these perpetuities cannot exist if $1+k \geq 1+i$.

Example 26.8
What is the present value of a stream of annual dividends, which starts at 1 at the end of the first year, and grows at the annual rate of 2%, given that the rate of interest is 6% ?

Solution.
The present value is $\frac{1}{i-k} = \frac{1}{0.06-0.02} = 25$ ■

Practice Problems

Problem 26.1
Smith receives \$400 in 1 year, \$800 in 2 years, \$1,200 in 3 years and so on until the final payment of \$4,000. Using an effective interest rate of 6%, determine the present value of these payments at time 0.

Problem 26.2
Find an expression for the present value at time 0 of payments of \$55 at time 1 year, \$60 at time 2 years, \$65 at time 3 years and so on, up to the last payment at time 20 years.

Problem 26.3
Find an expression for the accumulated value at time 20 years of payments of \$5 at time 1 year, \$10 at time 2 years, \$15 at time 3 years, and so on, up to \$100 at time 20 years.

Problem 26.4
Find the present value of a perpetuity-immediate whose successive payments are $\$5, \$10, \$15, \cdots$ assuming an annual effective rate of interest of 5%.

Problem 26.5
An annuity pays \$100 at the end of one month. It pays \$110 at the end of the second month. It pays \$120 at the end of the third month. The payments continue to increase by \$10 each month until the last payment is made at the end of the 36th month. Find the present value of the annuity at 9% compounded monthly.

Problem 26.6
An annual annuity-immediate pays \$100 at the end of the first year. Each subsequent payment is 5% greater than the preceding payment. The last payment is at the end of the 20th year. Calculate the accumulated value at:
(a) an annual effective interest rate of 4%;
(b) an annual effective interest rate of 5%.

Problem 26.7
A perpetuity pays \$100 at the end of the first year. Each subsequent annual payment increases by \$50. Calculate the present value at an annual effective interest rate of 10%.

Problem 26.8
A small business pays you an annual profit at the end of each year for 20 years. The payments grow at an annual rate of 2.5%, the first payment is \$10,000. What is the present value of this stream of payments at the annual effective rate of interest 6%?

Problem 26.9
What is the present value of a 30 year immediate annuity at 10% interest where the first payment is \$100 and each payment thereafter is increased by 5%?

Problem 26.10
A perpetuity-immediate has annual payments of 1, 3, 5, 7, $\cdots$. If the present values of the 6th and 7th payments are equal, find the present value of the perpetuity.

Problem 26.11
If X is the present value of a perpetuity of 1 per year with the first payment at the end of the 2nd year and $20X$ is the present value of a series of annual payments 1, 2, 3, $\cdots$ with the first payment at the end of the 3rd year, find d.

Problem 26.12
There are two perpetuities-immediate. The first has level payments of p at the end of each year. The second is increasing such that the payments are $q, 2q, 3q, \cdots$. Find the rate of interest that will make the difference in present value between these perpetuities
a) zero;
b) a maximum.

Problem 26.13
An annuity pays 10 at the end of the first year, 20 at the end of the second year, 30 at the end of the third year, etc. The last payment is made at the end of the 12th year.
Calculate the accumulated value of this annuity immediately after the last payment using an annual effective rate of 4%.

Problem 26.14
A 20 year annuity-immediate pays $500 + 50t$ at the end of year t.
Calculate the present value of this annuity using an annual effective rate of 6%.

Problem 26.15
An annuity pays 10 at the end of year 2, and 9 at the end of year 4. The payments continue decreasing by 1 each two year period until 1 is paid at the end of year 20. Calculate the present value of the annuity at an annual effective interest rate of 5%.

Problem 26.16
Find an expression for the present value of a perpetuity under which a payment of 1 is made at the end of the 1st year, 2 at the end of the 2nd year, increasing until a payment of n is made at the end of the nth year, and thereafter payments are level at n per year forever.

Problem 26.17
Find an expression for the present value of an annuity-immediate where payments start at 1, increase by 1 each period up to a payment of n, and then decrease by 1 each period up to a final payment of 1.

Problem 26.18
An 11-year annuity has a series of payments 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1, with the first payment made at the end of the second year. The present value of this annuity is 25 at interest rate i.
A 12-year annuity has a series of payments 1, 2, 3, 4, 5, 6, 6, 5, 4, 3, 2, 1, with the first payment made at the end of the first year.
Calculate the present value of the 12-year annuity at interest rate i.

Problem 26.19
Olga buys a 5-year increasing annuity for X. Olga will receive 2 at the end of the first month, 4 at the end of the second month, and for each month thereafter the payment increases by 2 . The nominal interest rate is 9% convertible quarterly. Calculate X.

Problem 26.20
An annuity-immediate has semiannual payments of 800, 750, 700, $\cdots$, 350, at $i^{(2)} = 0.16$. If $a_{\overline{10}|0.08} = A$, find the present value of the annuity in terms of A.

Problem 26.21 ‡
A perpetuity costs 77.1 and makes annual payments at the end of the year. The perpetuity pays 1 at the end of year 2, 2 at the end of year 3,$\cdots$, n at the end of year $(n+1)$. After year $(n+1)$, the payments remain constant at n. The annual effective interest rate is 10.5%. Calculate n.

Problem 26.22 ‡
A perpetuity-immediate pays 100 per year. Immediately after the fifth payment, the perpetuity is exchanged for a 25-year annuity-immediate that will pay X at the end of the first year. Each subsequent annual payment will be 8% greater than the preceding payment. The annual effective rate of interest is 8%. Calculate X.

Problem 26.23 ‡
A perpetuity-immediate pays 100 per year. Immediately after the fifth payment, the perpetuity is exchanged for a 25-year annuity-immediate that will pay X at the end of the first year. Each subsequent annual payment will be 8% greater than the preceding payment.
Immediately after the 10th payment of the 25-year annuity, the annuity will be exchanged for a perpetuity-immediate paying Y per year.
The annual effective rate of interest is 8%. Calculate Y.

Problem 26.24 ‡
Mike buys a perpetuity-immediate with varying annual payments. During the first 5 years, the payment is constant and equal to 10. Beginning in year 6, the payments start to increase. For year 6 and all future years, the current year's payment is K% larger than the previous year's payment. At an annual effective interest rate of 9.2%, the perpetuity has a present value of 167.50. Calculate K, given $K < 9.2$.

Problem 26.25 ‡
A company deposits 1000 at the beginning of the first year and 150 at the beginning of each subsequent year into perpetuity.
In return the company receives payments at the end of each year forever. The first payment is 100. Each subsequent payment increases by 5%. Calculate the company's yield rate for this transaction.

Problem 26.26 ‡
Megan purchases a perpetuity-immediate for 3250 with annual payments of 130. At the same price and interest rate, Chris purchases an annuity-immediate with 20 annual payments that begin at amount P and increase by 15 each year thereafter. Calculate P.

Problem 26.27 ‡
The present value of a 25-year annuity-immediate with a first payment of 2500 and decreasing by 100 each year thereafter is X. Assuming an annual effective interest rate of 10%, calculate X.

Problem 26.28 ‡
The present value of a series of 50 payments starting at 100 at the end of the first year and increasing by 1 each year thereafter is equal to X. The annual effective rate of interest is 9%. Calculate X.

Problem 26.29 ‡
An annuity-immediate pays 20 per year for 10 years, then decreases by 1 per year for 19 years. At an annual effective interest rate of 6%, the present value is equal to X. Calculate X.

Problem 26.30 ‡
At an annual effective interest rate of i, the present value of a perpetuity-immediate starting with a payment of 200 in the first year and increasing by 50 each year thereafter is 46,530. Calculate i.

Problem 26.31 ‡
An insurance company has an obligation to pay the medical costs for a claimant. Average annual claims costs today are \$5,000, and medical inflation is expected to be 7% per year. The claimant is expected to live an additional 20 years. Claim payments are made at yearly intervals, with the first claim payment to be made one year from today. Find the present value of the obligation if the annual interest rate is 5%.

Problem 26.32 ‡
Sandy purchases a perpetuity-immediate that makes annual payments. The first payment is 100, and each payment thereafter increases by 10.
Danny purchases a perpetuity-due which makes annual payments of 180.
Using the same annual effective interest rate, $i > 0$, the present value of both perpetuities are equal.
Calculate i.

Problem 26.33 ‡
Joe can purchase one of two annuities:
Annuity 1: A 10-year decreasing annuity-immediate, with annual payments of $10, 9, 8, \cdots, 1$.
Annuity 2: A perpetuity-immediate with annual payments. The perpetuity pays 1 in year 1, 2 in year 2, 3 in year 3, $\cdots$, and 11 in year 11 . After year 11, the payments remain constant at 11.
At an annual effective interest rate of i, the present value of Annuity 2 is twice the present value of Annuity 1 . Calculate the value of Annuity 1.

Problem 26.34 ‡
Mary purchases an increasing annuity-immediate for 50,000 that makes twenty annual payments as follows:
i) $P, 2P, \cdots, 10P$ in years 1 through 10; and
ii) $10(1.05)P, 10(1.05)^2P, \cdots, 10(1.05)^{10}P$ in years 11 through 20.
The annual effective interest rate is 7% for the first 10 years and 5% thereafter. Calculate P.

Problem 26.35
Find the present value of an annuity-immediate such that payments start at 1, each payment thereafter increases by 1 until reaching 10, and then remain at that level until 25 payments in total are made.

Problem 26.36
Find an expression of the present value of an annuity that pays 10 at the end of the fifth year and decreases by 1 thereafter until reaching 0.

Problem 26.37
Annual deposits are made into a fund at the beginning of each year for 10 years. The first five deposits are \$1000 each and deposits increase by 5% per year thereafter. If the fund earns 8% effective, find the accumulated value at the end of 10 years.

Problem 26.38
Find the present value of a 20-year annuity-immediate with first payment of \$600 and each subsequent payment is 5% greater than the preceding payment. The annual effective rate of interest is 10.25%.

Problem 26.39
A perpetuity pays 200 at the end of each of the first two years, 300 at the end of years 3 and 4, 400 at the end of years 5 and 6, etc. Calculate the present value of this perpetuity if the annual effective rate of interest is 10%.

Problem 26.40
Janice is receiving a perpetuity of 100 at the end of each year. Megan is receiving a perpetuity that pays 10 at the end of the first year, 20 at the end of the second year, 30 at the end of the third year, etc. The present values of the two perpetuities are equal. Calculate the annual effective interest rate used to determine the present values.

Problem 26.41
Gloria borrows 100,000 to be repaid over 30 years. You are given
(1) Her first payment is X at the end of year 1
(2) Her payments increase at the rate of 100 per year for the next 19 years and remain level for the following 10 years
(3) The effective rate of interest is 5% per annum
Calculate X.

Problem 26.42
You are given a perpetual annuity immediate with annual payments increasing in geometric progression, with a common ratio of 1.07. The annual effective interest rate is 12%. The first payment is 1. Calculate the present value of this annuity.

Problem 26.43
An annuity immediate pays 10 at the ends of years 1 and 2, 9 at the ends of years 3 and 4, etc., with payments decreasing by 1 every second year until nothing is paid. The effective annual rate of interest is 5%. Calculate the present value of this annuity immediate.

Problem 26.44
Barbara purchases an increasing perpetuity with payments occurring at the end of every 2 years. The first payment is 1, the second one is 2, the third one is 3, etc. The price of the perpetuity is 110. Calculate the annual effective interest rate.

Problem 26.45
You are given $(Ia)_{\overline{n-1}|} = \frac{K}{d}$, where d is the annual effective discount rate. Calculate K.

Problem 26.46
Francois purchases a 10 year annuity immediate with annual payments of $10X$. Jacques purchases

a 10 year decreasing annuity immediate which also makes annual payments. The payment at the end of year 1 is equal to 50. At the end of year 2, and at the end of each year through year 10, each subsequent payment is reduced over what was paid in the previous year by an amount equal to X. At an annual effective interest rate of 7.072%, both annuities have the same present value. Calculate X, where $X < 5$.

Problem 26.47 ‡
1000 is deposited into Fund X, which earns an annual effective rate of 6%. At the end of each year, the interest earned plus an additional 100 is withdrawn from the fund. At the end of the tenth year the fund is depleted. The annual withdrawals of interest and principal are deposited into Fund Y, which earns an annual effective rate of 9%. Determine the accumulated value of Fund Y at the end of year 10.

Problem 26.48
You are given two series of payments. Series A is a perpetuity with payments of 1 at the end of each of the first 2 years, 2 at the end of each of the next 2 years, 3 at the end of each of the next 2 years, and so on. Series B is a perpetuity with payments of K at the end of each of the first 3 years, $2K$ at the end of the next 3 years, $3K$ at the end of each of the next 3 years, and so on. The present values of the two series of payments are equal.
Calculate K.

Problem 26.49
John deposits 100 at the end of each year for 20 years into a fund earning an annual effective interest rate of 7%.
Mary makes 20 deposits into a fund at the end of each year for 20 years. The first 10 deposits are 100 each, while the last 10 deposits are $100 + X$ each. The fund earns an annual effective interest rate of 8% during the first 10 years and 6% annual effective interest thereafter.
At the end of 20 years, the amount in John's fund equals the amount in Mary's fund.
Calculate X.

Problem 26.50
An annuity-immediate pays an initial benefit of 1 per year, increasing by 10.25% every four years. The annuity is payable for 40 years. Using an annual effective interest rate of 5%, determine an expression for the present value of this annuity.

Problem 26.51
An increasing perpetuity with annual payments has a present value of 860 at an annual effective discount rate, d. The initial payment at the end of the first year is 3 and each subsequent payment is 2 more than its preceding payment. Calculate d.

Problem 26.52
A 10-year increasing annuity-immediate paying 5 in the first year and increasing by 5 each year thereafter has a present value of G. A 10-year decreasing annuity-immediate paying y in the first year and decreasing by $\frac{y}{10}$ each year thereafter has a present value of G. Both present values are calculated using an annual effective interest rate of 4%. Calculate y to the nearest integer.

Problem 26.53
A loan of 10,000 is being repaid by 10 semiannual payments, with the first payment made one-half year after the loan. The first 5 payments are K each, and the final 5 are $K + 200$ each. Given $i^{(2)} = 0.6$, determine K.

Problem 26.54
An annuity immediate has the following payment pattern: $1, 2, \cdots, n-1, n, n-1, \cdots, 2, 1$. The present value of this annuity using an annual effective interest rate of 9% is 129.51. Determine n.

Problem 26.55
An increasing 25-year annuity immediate has an initial payment in year one of 4 and each subsequent annual payment is 3 more than the preceding one. Find the present value of this annuity using an annual effective interest rate of 7%.

Problem 26.56
A 10-year annuity-immediate has a first payment of 2,000. Each subsequent annual payment is 100 less than the preceding payment. (The payments are 2,000; 1,900; 1,800; etc.)
At an annual effective interest rate of 5%, what is the accumulated value of this stream of payments on the date of the final payment?

Problem 26.57
A payment of 100 purchases an increasing perpetuity with a first payment of 10 at the end of the first year. Each annual payment thereafter is 5% greater than the prior year's payment.
What is the annual interest rate for this stream of payments?

Problem 26.58
You are given a perpetuity, with annual payments as follows:
(1) Payments of 1 at the end of the first year and every three years thereafter.
(2) Payments of 2 at the end of the second year and every three years thereafter.
(3) Payments of 3 at the end of the third year and every three years thereafter.
(4) The interest rate is 5% convertible semi-annually.
Calculate the present value of this perpetuity.

Problem 26.59
(a) You just paid \$50 for a share of stock which pays annual dividends. You expect to receive the first dividend payment of \$5 one year from now. After that, you expect dividends to increase by \$X each year, forever. The price you paid implies that you would be happy with an annual return of 12%. What is X?
(b) What value \$Y could you sell your share for if the new buyer has the same desired annual return but expects the annual dividend increase to be $X + 1$?

Problem 26.60
A 30-year annuity-immediate pays 20,000 in the first year. Thereafter, the payments decrease 500 each year. If $d^{(4)}$ is 6%, what is the present value of this annuity?

Problem 26.61
You are about to buy shares in a company. The first annual dividend, of \$3 per share, is expected to be paid 3 years from now. Thereafter, you expect that the dividend will grow at the rate of 10% per year for ever. Assuming your desired return is 12%, what are you willing to pay per share for this stock?

Problem 26.62
A perpetuity with annual payments is payable beginning 10 years from now. The first payment is 50. Each annual payment thereafter is increased by 10 until a payment of 150 is reached. Subsequent payments remain level at 150. This perpetuity is purchased by means of 10 annual premiums, with the first premium of P due immediately. Each premium after the first is 105% of the preceding one. The annual effective interest rates are 5% during the first 9 years and 3% thereafter. Calculate P.

Problem 26.63
Show that $(Da)_{\overline{n|}} = \sum_{t=1}^{n} a_{\overline{t|}}$.

Problem 26.64
Simplify $\sum_{t=1}^{10}(t+1)\nu^t$.

27 Varying Annuity-Due

In this section, we examine the case of an increasing annuity-due. Consider an annuity with the first payment is P at the beginning of year 1 and then the payments increase by Q thereafter, continuing for n years. A time diagram of this situation is given in Figure 27.1

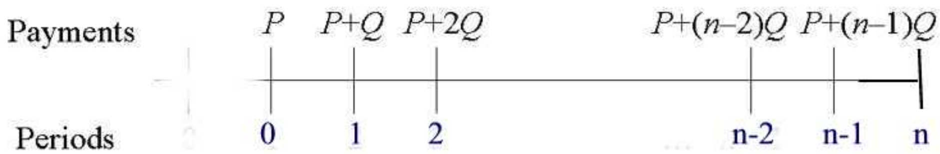


Figure 27.1

The present value for this annuity-due is

$$PV = P + (P+Q)\nu + (P+2Q)\nu^2 + \cdots + [P+(n-1)Q]\nu^{n-1}. \tag{27.1}$$

Multiplying Equation (28.1) by ν, we obtain

$$\nu PV = P\nu + (P+Q)\nu^2 + (P+2Q)\nu^3 + \cdots + [P+(n-1)Q]\nu^n. \tag{27.2}$$

Subtracting Equation(28.2) from Equation (28.1) we obtain

$$(1-\nu)PV = P(1-\nu^n) + (\nu + \nu^2 + \cdots + \nu^n)Q - n\nu^n Q$$

or

$$PV = P\ddot{a}_{\overline{n}|} + Q\frac{[a_{\overline{n}|} - n\nu^n]}{d}.$$

The accumulated value of these payments at time n is

$$AV = (1+i)^n PV = P\ddot{s}_{\overline{n}|} + Q\frac{[s_{\overline{n}|} - n]}{d}.$$

In the special case when $P = Q = 1$ we find

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{d}$$

and

$$(I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d}.$$

Example 27.1
Determine the present value and future value of payments of \$75 at time 0, \$80 at time 1 year, \$85 at time 2 years, and so on up to \$175 at time 20 years. The annual effective rate is 4%.

Solution.
The present value is $70\ddot{a}_{\overline{21}|} + 5(I\ddot{a})_{\overline{21}|} = \$1,720.05$ and the future value is $(1.04)^{21}(1,720.05) = \$3919.60$ ■

In the case of a decreasing annuity-due where $P = n$ and $Q = -1$, the present value at time 0 is

$$(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}$$

and the accumulated value at time n is

$$(D\ddot{s})_{\overline{n}|} = (1+i)^n(D\ddot{a})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}.$$

Example 27.2
Calculate the present value and the accumulated value of a series of payments of \$100 now, \$90 in 1 year, \$80 in 2 years, and so on, down to \$10 at time 9 years using an annual effective interest rate of 3%.

Solution.
The present value is $10(D\ddot{a})_{\overline{10}|} = 10\cdot\frac{10-8.530203}{0.03/1.03} = \$504.63$ and the accumulated value is $(1.03)^{10}(504.63) = \$678.18$ ■

Next, we consider a perpetuity-due with payments that form an arithmetic progression (and of course $P > 0$ and $Q > 0$). The present value for such a perpetuity with the first payment at time 0 is

$$\begin{aligned}
PV = &\lim_{n\to\infty}\left[P\ddot{a}_{\overline{n}|} + Q\frac{[a_{\overline{n}|} - n\nu^n]}{d}\right]\\
=&P\lim_{n\to\infty}\ddot{a}_{\overline{n}|} + Q\lim_{n\to\infty}\frac{[a_{\overline{n}|} - n\nu^n]}{d}\\
=&P\lim_{n\to\infty}\ddot{a}_{\overline{n}|} + Q\frac{[\lim_{n\to\infty} a_{\overline{n}|} - \lim_{n\to\infty} n\nu^n]}{d}\\
=&P\ddot{a}_{\overline{\infty}|} + \frac{Qa_{\overline{\infty}|}}{d} = \frac{P}{d} + \frac{Q(1+i)}{i^2}
\end{aligned}$$

since $a_{\overline{\infty}|} = \frac{1}{i},\ \ddot{a}_{\overline{\infty}|} = \frac{1}{d}$ and $\lim_{n\to\infty} n\nu^n = 0$ (by L'Hopital's rule).
In the special case $P = Q = 1$ we find

$$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}.$$

Example 27.3
Determine the present value at time 0 of payments of \$10 paid at time 0, \$20 paid at time 1 year, \$30 paid at time 2 years, and so on, assuming an annual effective rate of 5%.

Solution.
The answer is $10(I\ddot{a})_{\overline{\infty}|} = \frac{10}{d^2} = 10\left(\frac{1.05}{0.05}\right)^2 = \$4,410.00$ ■

Next, we consider payments varying in a geometric progression. Consider an annuity-due with a term of n periods where the interest rate is i per period, and where the first payment is 1 at time 0 and successive payments increase in geometric progression with common ratio $1+k$. The present value of this annuity is

$$\begin{aligned}1 + \nu(1+k) + \nu^2(1+k)^2 + \cdots + \nu^{n-1}(1+k)^{n-1} =&\frac{1-[(1+k)\nu]^n}{1-(1+k)\nu}\\ =&\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\left(\frac{1+k}{1+i}\right)}\\ =&(1+i)\frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k}\end{aligned}$$

provided that $k \neq i$. If $k = i$ then the original sum consists of a sum of n terms of 1 which equals to n.

Example 27.4
An annual annuity due pays \$1 at the beginning of the first year. Each subsequent payment is 5% greater than the preceding payment. The last payment is at the beginning of the 10th year. Calculate the present value at:
(a) an annual effective interest rate of 4%;
(b) an annual effective interest rate of 5%.

Solution.
(a) $PV = (1.04)\frac{\left[1-\left(\frac{1.05}{1.04}\right)^{10}\right]}{(0.04-0.05)} = \$10.44.$
(b) Since $i = k$, $PV = n = \$10.00$ ■

For an annuity-due with n payments where the first payment is 1 at time 0 and successive payments decrease in geometric progression with common ratio $1-k$. The present value of this annuity is

$$\begin{aligned}1+\nu(1-k)+\nu^2(1-k)^2+\cdots+\nu^{n-1}(1-k)^{n-1} =&\frac{1-[(1-k)\nu]^n}{1-(1-k)\nu}\\ =&\frac{1-\left(\frac{1-k}{1+i}\right)^n}{1-\left(\frac{1-k}{1+i}\right)}\\ =&(1+i)\frac{1-\left(\frac{1-k}{1+i}\right)^n}{i+k}\end{aligned}$$

provided that $k\neq i$. If $k=i$ then the original sum is

$$1+\nu(1-i)+\nu^2(1-i)^2+\cdots+\nu^{n-1}(1-i)^{n-1}=\frac{1}{2d}\left[1-\left(\frac{1-i}{1+i}\right)^n\right].$$

Example 27.5 ‡
Matthew makes a series of payments at the beginning of each year for 20 years. The first payment is 100. Each subsequent payment through the tenth year increases by 5% from the previous payment. After the tenth payment, each payment decreases by 5% from the previous payment. Calculate the present value of these payments at the time the first payment is made using an annual effective rate of 7%.

Solution.
The present value at time 0 of the first 10 payments is

$$100\left[\frac{1-\left(\frac{1.05}{1.07}\right)^{10}}{0.07-0.05}\right]\cdot(1.07)=919.95.$$

The value of the 11th payment is $100(1.05)^9(0.95)=147.38$. The present value of the last ten payments is

$$147.38\left[\frac{1-\left(\frac{0.95}{1.07}\right)^{10}}{0.07+0.05}\right]\cdot(1.07)(1.07)^{-10}=464.71.$$

The total present value of the 20 payments is $919.95+464.71=1384.66$ ■

Finally, the present value of a perpetuity with first payment of 1 at time 0 and successive payments increase in geometric progression with common ration $1+k$ is

$$1+\nu(1+k)+\nu^2(1+k)^2+\cdots=\frac{1}{1-(1+k)\nu}=\frac{1+i}{i-k}.$$

Observe that the value for these perpetuities cannot exist if $1+k\geq 1+i$.

Example 27.6
Perpetuity A has the following sequence of annual payments beginning on January 1, 2005:

$$1, 3, 5, 7, \cdots$$

Perpetuity B is a level perpetuity of 1 per year, also beginning on January 1, 2005.
Perpetuity C has the following sequence of annual payments beginning on January 1, 2005:

$$1, 1+r, (1+r)^2, \cdots$$

On January 1, 2005, the present value of Perpetuity A is 25 times as large as the present value of Perpetuity B, and the present value of Perpetuity A is equal to the present value of Perpetuity C. Based on this information, find r.

Solution.
The present value of Perpetuity A is $\frac{1}{d} + \frac{2(1+i)}{i^2}$.
The present value of Perpetuity B is $\frac{1}{d}$.
The present value of Perpetuity C is $\frac{1+i}{i-r}$.
We are told that

$$\frac{1+i}{i} + \frac{2(1+i)}{i^2} = \frac{25(1+i)}{i}.$$

This is equivalent to

$$12i^2 + 11i - 1 = 0.$$

Solving for i we find $i = \frac{1}{12}$. Also, we are told that

$$\frac{1+i}{i} + \frac{2(1+i)}{i^2} = \frac{1+i}{i-r}$$

or

$$25(12)(1 + \frac{1}{12}) = \frac{1 + \frac{1}{12}}{\frac{1}{12} - r}.$$

Solving for r we find $r = 0.08 = 8\%$ ■

Practice Problems

Problem 27.1
A 20 year increasing annuity due pays 100 at the start of year 1, 105 at the start of year 2, 110 at the start of year 3, etc. In other words, each payment is 5% greater than the prior payment. Calculate the present value of this annuity at an annual effective rate of 5%.

Problem 27.2
A perpetuity pays 1000 at the beginning of the first year. Each subsequent payment is increased by inflation. If inflation is assumed to be 4% per year, calculate the present value of this perpetuity using an annual effective interest rate of 10%.

Problem 27.3
A perpetuity pays 1000 at the beginning of the first year. Each subsequent payment is increased by inflation. If inflation is assumed to be 4% per year, calculate the present value of this perpetuity using an annual effective interest rate of 4%.

Problem 27.4
A 10-year decreasing annuity-due makes a payment of 100 at the beginning of the first year, and each subsequent payment is 5 less then the previous. What is the accumulated value of this annuity at time 10 (one year after the final payment)? The effective annual rate of interest is 5%.

Problem 27.5
A 10 year annuity due pays 1,000 as the first payment. Each subsequent payment is 50 less than the prior payment.
Calculate the current value of the annuity at the end of 5 years using an annual effective rate of 8%.

Problem 27.6
You buy an increasing perpetuity-due with annual payments starting at 5 and increasing by 5 each year until the payment reaches 100. The payments remain at 100 thereafter. The annual effective interest rate is 7.5%. Determine the present value of this perpetuity.

Problem 27.7
Debbie receives her first annual payment of 5 today. Each subsequent payment decreases by 1 per year until time 4 years. After year 4, each payment increases by 1 until time 8 years. The annual interest rate is 6%. Determine the present value.

Problem 27.8
Determine the accumulated value at time 20 years of payments of \$10 at time 0, \$20 at time 1 year, \$30 at time 2 years, and so on, up to \$200 at time 19 years. The annual effective rate of interest is 4%.

Problem 27.9
Perpetuity X has payments of $1, 2, 3, \cdots$ at the beginning of each year. Perpetuity Y has payments of $q, q, 2q, 2q, 3q, 3q, \cdots$ at the beginning of each year. The present value of X is equal to the present value of Y at an effective annual interest rate of 10%. Calculate q.

Problem 27.10
Kendra receives \$900 now, \$970 in 1 year, \$1040 in 2 years, \$1,110 in 3 years, and so on, until the final payment of \$1600. Using an annual effective rate of interest of 9%, find
(a) the present value of these payments at time 0;
(b) the accumulated value at time 11 years.

Problem 27.11
Mary is saving money for her retirement. She needs \$750,000 in 10 years to purchase a retirement apartment in Florida. She invests X, now, $X - 5,000$ in 1 year, $X - 10,000$ in 2 years, and so on, down to $X - 45,000$ in 9 years. Using an annual effective rate of interest of 5%, find X.

Problem 27.12
Chris makes annual deposits into a bank account at the beginning of each year for 20 years. Chris' initial deposit is equal to 100, with each subsequent deposit k% greater than the previous year's deposit. The bank credits interest at an annual effective rate of 5%. At the end of 20 years, the accumulated amount in Chris' account is equal to 7276.35. Given $k > 5$, calculate k.

Problem 27.13
You are given:
(i) The present value of an annuity-due that pays 300 every 6 months during the first 15 years and 200 every 6 months during the second 15 years is 6000.
(ii) The present value of a 15-year deferred annuity-due that pays 350 every 6 months for 15 years is 1580.
(iii) The present value of an annuity-due that pays 100 every 6 months during the first 15 years and 200 every 6 months during the next 15 years is X.
The same interest rate is used in all calculations. Determine X.

Problem 27.14
A 20 year annuity-due with annual payments has a first payment of 100 and each subsequent annual payment is 5% more than its preceding payment. Calculate the present value of this annuity at a nominal interest rate of 10% compounded semiannually.

Problem 27.15
The payments of a perpetuity-due are three years apart. The initial payment is 7 and each subsequent payment is 5 more than its previous payment. Find the price (present value at time 0) of this perpetuity-due, using an annual effective discount rate of 6%.

Problem 27.16
You are considering the purchase of a share of XYZ stock. It pays a quarterly dividend. You expect to receive the first dividend, of \$2.50, 3 months from now. You expect the dividend to increase by 6% per year, forever, starting with the 5th dividend. If the desired return on your investment is 15%, what is the amount your are willing to pay for a share of XYZ stock?

Problem 27.17
You plan to make annual deposits at the start of each year for 25 years into your retirement fund. The first 5 deposits are to be 2,000 each. Thereafter, you plan to increase the deposits by X per year. Assuming the fund can earn an effective annual interest rate of 6%, what must X be for you to have accumulated 250,000 at the end of the 25 years?

28 Varying Annuities with Payments at a Different Frequency than Interest is Convertible

In Sections 26 and 27, we discussed varying annuities where the payment period coincide with the interest conversion period. In this section we consider varying annuities with payments made more or less frequently than interest is convertible. We will limit our discussion to increasing annuities. Decreasing annuities can be handled in a similar fashion.

Varying Annuities Payable Less Frequently Than Interest is Convertible
Consider annuities payable less frequently than interest is convertible. We let k be the number of interest conversion periods in one payment period, n the term of the annuity measured in interest conversion periods, and i the rate of interest per conversion period. It follows that the number of payments over the term of the annuity is given by $\frac{n}{k}$ which we assume is a positive integer.
Let PV be the present value of a generalized increasing annuity-immediate with payments 1, 2, 3, etc. occurring at the end of each interval of k interest conversion periods. Then,

$$PV = \nu^k + 2\nu^{2k} + 3\nu^{3k} + \cdots + \left(\frac{n}{k} - 1\right)\nu^{n-k} + \frac{n}{k}\nu^n. \tag{28.1}$$

Multiply Equation (28.1) by $(1+i)^k$ to obtain

$$(1+i)^k PV = 1 + 2\nu^k + 3\nu^{2k} + \cdots + \left(\frac{n}{k} - 1\right)\nu^{n-2k} + \frac{n}{k}\nu^{n-k}. \tag{28.2}$$

Now subtracting Equation (28.1) from Equation (28.2) we find

$$PV[(1+i)^k - 1] = 1 + \nu^k + \nu^{2k} + \cdots + \nu^{n-k} - \frac{n}{k}nu^n = \frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k}\nu^n.$$

Hence,

$$PV = \frac{\frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k}\nu^n}{is_{\overline{k}|}}.$$

The accumulated value at time $t = n$ is

$$AV = (1+i)^n PV = (1+i)^n \frac{\frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k}\nu^n}{is_{\overline{k}|}}.$$

Example 28.1
A 10-year annuity-immediate pays 1 in two years, 2 in four years, 3 in six years, 4 in eight years, and 5 in ten years. Using an effective annual interest rate 5%, find the present value and the accumulated value of this annuity.

Solution.
The present value of this annuity at time 0 year is

$$PV = \frac{\frac{a_{\overline{10}|}}{a_{\overline{2}|}} - 5(1.05)^{-10}}{0.05s_{\overline{2}|}} = \frac{\frac{7.721735}{1.85941} - 5(1.05)^{-10}}{(1.05)^2 - 1} = 10.568.$$

The accumulated value is $10.568(1.05)^{10} = \$17.21$ ■

Now, consider an increasing annuity-due with payments 1, 2, 3, etc. occurring at the start of each interval of k interest conversion periods. Then

$$PV = 1 + 2\nu^k + 3\nu^{2k} + \cdots + \left(\frac{n}{k} - 1\right)\nu^{n-2k} + \frac{n}{k}\nu^{n-k}. \quad (28.3)$$

Multiply Equation (28.3) by ν^k to obtain

$$\nu^k PV = \nu^k + 2\nu^{2k} + 3\nu^{3k} + \cdots + \left(\frac{n}{k} - 1\right)\nu^{n-k} + \frac{n}{k}\nu^n. \quad (28.4)$$

Now subtracting Equation (28.4) from Equation (28.3) we find

$$PV[1 - \nu^k] = 1 + \nu^k + \nu^{2k} + \cdots + \nu^{n-k} - \frac{n}{k}\nu^n = \frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k}\nu^n.$$

Hence,

$$PV = \frac{\frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k}\nu^n}{ia_{\overline{k}|}}.$$

The accumulated value at time $t = n$ is

$$AV = (1+i)^n \frac{\frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k}\nu^n}{ia_{\overline{k}|}}.$$

Example 28.2
Find the present value and the accumulated value of a 5-year annuity in which payments are made at the beginning of each half-year, with the first payment of \$50, the second payment of \$100, the third payment of \$150, and so on. Interest is 10% convertible quarterly.

Solution.
The present value is

$$PV = 50\frac{\frac{a_{\overline{20}|}}{a_{\overline{2}|}} - 10\nu^{20}}{0.025a_{\overline{2}|}} = \frac{\frac{15.58916}{1.92742} - 10(1.025)^{-20}}{1 - (1.025)^{-2}} = \$2060.15.$$

The accumulated value at time $t = 20$ is $(2060.15)(1.025)^{20} = \$3375.80$ ■

Varying Annuities Payable More Frequently Than Interest is Convertible
We next consider increasing annuities where payments are made more frequently than interest is convertible. We examine two different types. The first is the increasing $m$thly annuity where the increase occurs once per conversion period. The second is the $m$thly increasing $m$thly annuity where the increase takes place with each $m$thly payment.

Annuities Payable $m$thly
Consider first an increasing annuity-immediate with constant payments during each interest conversion period with increases occurring only once per interest conversion period. Let the annuity be payable $m$thly with each payment in the first period equal to $\frac{1}{m}$, each payment in the second period equal to $\frac{2}{m}, \cdots$, each payment in the nth period equal to $\frac{n}{m}$ as shown in Figure 28.1.

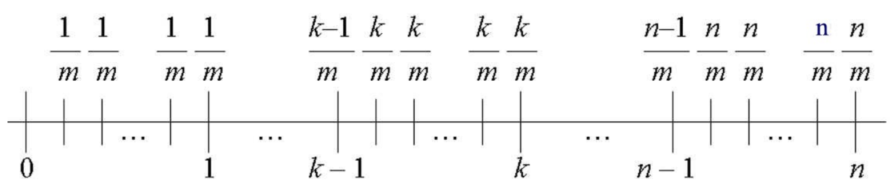


Figure 28.1

Let i be the rate per one conversion period and $i^{(m)}$ be the nominal interest rate payable m times per interest conversion period. Making the $m$thly payments of $\frac{k}{m}$ in period k is the same as making one payment equal to the accumulated value of the $m$thly payments at the end of the period, that is, one payment of

$$\frac{k}{m} s_{\overline{m}|j} = \frac{k}{m} \cdot \frac{(1+j)^m - 1}{j} = k \frac{i}{i^{(m)}}$$

where $j = \frac{i^{(m)}}{m}$. The time diagram of this is given in Figure 28.2.

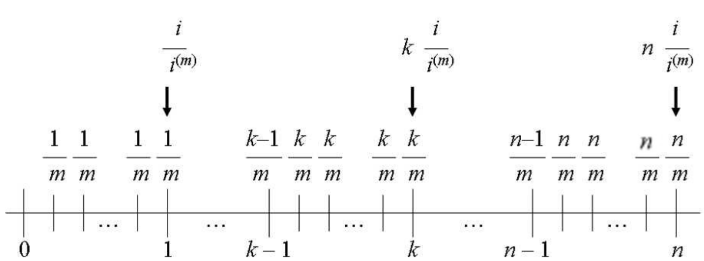


Figure 28.2

Consequently, the increasing annuity payable $m$thly is the same as an increasing annuity with $P = Q = \frac{i}{i^{(m)}}$ which implies the present value

$$(Ia)^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}}(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{i^{(m)}}.$$

The accumulated value is

$$(Is)^{(m)}_{\overline{n}|} = (1+i)^n(Ia)^{(m)}_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i^{(m)}}.$$

Example 28.3
Determine the present value and the accumulated value of 1 at the end of each quarter in the first year, 2 at the end of each quarter in the second year, 3 at the end of each quarter in the third year, 4 at the end of each quarter in the fourth year, and 5 at the end of each quarter in the fifth year. The annual effective interest rate is 4%

Solution.
The factor $(Ia)^{(4)}_{\overline{5}|}$ values a payment of 1/4 at the end of each quarter during the first year, 2/4 at the end of each quarter during the second year, and so on. So, the present value is

$$4(Ia)^{(4)}_{\overline{5}|} = 4 \cdot \frac{\ddot{a}_{\overline{5}|} - 5\nu^5}{i^{(4)}}.$$

But

$$\begin{aligned} i^{(4)} =& 4[(1.04)^{\frac{1}{4}} - 1] = 3.9414\% \\ \ddot{a}_{\overline{5}|} =& \frac{1-(1.04)^{-5}}{0.04/1.04} = 4.629895 \\ (Ia)^{(4)}_{\overline{5}|} =& \frac{4.629895 - 5(1.04)^{-5}}{0.039414} = 13.199996 \\ 4(Ia)^{(4)}_{\overline{5}|} =& 52.80. \end{aligned}$$

The accumulated value is

$$(Is)^{(4)}_{\overline{5}|} = (1.04)^5(Ia)^{(4)}_{\overline{5}|} = 1.04^5 \times 52.80 = 64.24 \blacksquare$$

In the case of an annuity-due, one can easily show that

$$(I\ddot{a})^{(m)}_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n\nu^n}{d^{(m)}}$$

and

$$(I\ddot{s})^{(m)}_{\overline{n}|} = (1+i)^n(I\ddot{a})^{(m)}_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d^{(m)}}.$$

Example 28.4
Determine the present value and accumulated value of 10 at the start of each quarter in the first year, 20 at the start of each quarter in the second year, and so on for 5 years. The annual effective interest rate is 4%.

Solution.
The present value is

$$\begin{aligned}40(I\ddot{a})_{\overline{5}|}^{(4)} =&40 \times \frac{\ddot{a}_{\overline{5}|} - 5\nu^5}{d^{(4)}}\\ =&40 \times \frac{4.629895 - 5(1.04)^{-5}}{0.039029}\\ =&533.20\end{aligned}$$

where

$$d^{(4)} = 4[1 - (1.04)^{-\frac{1}{4}}] = 3.9029\%.$$

The accumulated value is

$$(1.04)^5 \times 533.20 = 648.72 \blacksquare$$

$m$thly Increasing $m$thly Payable Annuities
Consider next a situation in which payments vary within each interest conversion period as shown in Figure 28.3. Such an annuity is referred to as **$m$thly increasing $m$thly annuity**.

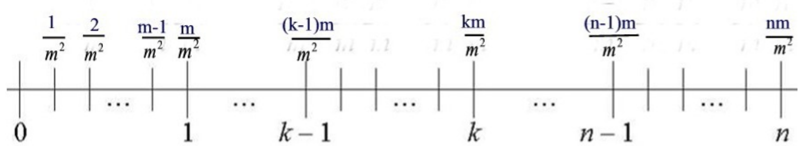


Figure 28.3

Denoting the present value of such an annuity by $(I^{(m)}a)_{\overline{n}|}^{(m)}$ we find

$$(I^{(m)}a)_{\overline{n}|}^{(m)} = \frac{1}{m^2}[\nu^{\frac{1}{m}} + 2\nu^{\frac{2}{m}} + \cdots + mn\nu^{\frac{mn}{m}}]. \tag{28.5}$$

Multiply both sided by $(1+i)^{\frac{1}{m}}$ to obtain

$$(1+i)^{\frac{1}{m}}(I^{(m)}a)_{\overline{n}|}^{(m)} = \frac{1}{m^2}[1 + 2\nu^{\frac{1}{m}} + \cdots + mn\nu^{n-\frac{1}{m}}]. \tag{28.6}$$

Subtract Equation (28.5) from Equation (28.6) to obtain

$$\begin{aligned}(I^{(m)}a)_{\overline{n}|}^{(m)}[(1+i)^{\frac{1}{m}}-1]=&\frac{1}{m^2}[1+\nu^{\frac{1}{m}}+\cdots+\nu^{n-\frac{1}{m}}-nm\nu^n]\\=&\frac{1}{m}[\ddot{a}_{\overline{n}|}^{(m)}-n\nu^n]\\(I^{(m)}a)_{\overline{n}|}^{(m)}=&\frac{\ddot{a}_{\overline{n}|}^{(m)}-n\nu^n}{m[(1+i)^{\frac{1}{m}}-1]}\\=&\frac{\ddot{a}_{\overline{n}|}^{(m)}-n\nu^n}{i^{(m)}}\end{aligned}$$

where

$$\ddot{a}_{\overline{n}|}^{(m)}=\frac{1-\nu^n}{d^{(m)}}=\frac{i}{d^{(m)}}a_{\overline{n}|}.$$

The accumulated value is

$$(I^{(m)}s)_{\overline{n}|}^{(m)}=(1+i)^n(I^{(m)}a)_{\overline{n}|}^{(m)}=\frac{\ddot{s}_{\overline{n}|}^{(m)}-n}{i^{(m)}}$$

where

$$\ddot{s}_{\overline{n}|}^{(m)}=\frac{(1+i)^m-1}{d^{(m)}}.$$

Example 28.5
Determine the present value and the accumulated value of 1 at the end of the first quarter, 2 at the end of the second quarter, 3 at the end of the third quarter, and so on for 5 years. The annual effective interest rate is 4%.

Solution.
The factor $(I^{(4)}a)_{\overline{5}|}^{(4)}$ values a payment of $\frac{1}{16}$ at the end of the first quarter, $\frac{2}{16}$ at the end of the second quarter, and so on. So the present value is

$$16(I^{(4)}a)_{\overline{5}|}^{(4)}=16\times\frac{\ddot{a}_{\overline{5}|}^{(4)}-5\nu^5}{i^{(4)}}=16\times\frac{4.56257-5(1.04)^{-5}}{0.039414}=183.87$$

and the accumulated value is

$$(1.04)^5\times 183.87=223.71\ \blacksquare$$

In the case of an annuity-due, it can be easily shown that

$$(I^{(m)}\ddot{a})_{\overline{n}|}^{(m)}=\frac{\ddot{a}_{\overline{n}|}^{(m)}-n\nu^n}{d^{(m)}}$$

and

$$(I^{(m)}\ddot{s})_{\overline{n}|}^{(m)} = (1+i)^n (I^{(m)}\ddot{a})_{\overline{n}|}^{(m)} = \frac{\ddot{s}_{\overline{n}|}^{(m)} - n}{d^{(m)}}.$$

Example 28.6
Determine the present value and the accumulated value of 1 at the start of the first quarter, 2 at the start of the second quarter, 3 at the start of the third quarter, and so on for 5 years. The annual effective interest rate is 4%.

Solution.
The present value is

$$16(I^{(5)}\ddot{a})_{\overline{5}|}^{(4)} = 16 \times \frac{\ddot{a}_{\overline{5}|}^{(4)} - 5\nu^5}{d^{(4)}} = \frac{4.56257 - 5(1.04)^{-5}}{0.03903} = 185.68$$

and the accumulated value is

$$(1.04)^5 \times 185.68 = 225.91 \blacksquare$$

Other Considerations
Types of annuities like the one discussed above but that vary in geometric progression can be handled by expressing the annuity value as a summation of the present value or accumulated value of each payment. This summation is a geometric progression which can be directly evaluated. We illutrate this in the next example.

Example 28.7
Find the accumulated value at the end of eight years of an annuity in which payments are made at the beginning of each quarter for four years. The first payment is $3,000, and each of the other payments is 95% of the previous payment. Interest is credited at 6% convertible semiannually.

Solution.
We count periods in quarters of a year. If j is the interest rate per quarter then $1+j = (1.03)^{\frac{1}{2}}$. The accumulated value at $t = 8 \times 4 = 32$ is

$$\begin{aligned}
&3000[(1.03)^{16} + (0.95)(1.03)^{15.5} + (0.95)^2(1.03)^{15} + \cdots + (0.95)^{15}(1.03)^{8.5}]\\
=&3000(1.03)^{16}\left[1 + \left(\frac{0.95}{\sqrt{1.03}}\right) + \left(\frac{0.95}{\sqrt{1.03}}\right)^2 + \cdots + \left(\frac{0.95}{\sqrt{1.03}}\right)^{15}\right]\\
=&3000(1.03)^{16} \cdot \frac{1 - \left(\frac{0.95}{\sqrt{1.03}}\right)^{16}}{1 - \left(\frac{0.95}{\sqrt{1.03}}\right)} = \$49,134.18 \blacksquare
\end{aligned}$$

Example 28.8
Find an expression for the present value of a perpetuity which pays 1 at the end of the third year, 2 at the end of the sixth year, 3 at the end of the ninth year and so on.

Solution.
The present value satisfies the equation

$$PV = \nu^3 + 2\nu^6 + 3\nu^9 + \cdots$$

Thus,

$$PV(1-\nu^3) = \nu^3 + \nu^6 + \nu^9 + \cdots = \frac{\nu^3}{1-\nu^3}$$

or

$$PV = \frac{\nu^3}{(1-\nu^3)^2} \blacksquare$$

Some of the problems can be solved using the basic principles.

Example 28.9
Scott deposits 1 at the beginning of each quarter in year 1, 2 at the beginning of each quarter in year 2, $\cdots$, 8 at the beginning of each quarter in year 8. One quarter after the last deposit Scott withdraws the accumulated value of the fund and uses it to buy a perpetuity immediate with level payments of X at the end of each year. All calculations assume a nominal interest rate of 10% per annum compounded quarterly. Calculate X.

Solution.
Regard the annuity as an annuity-due with 8 annual payments: $k\ddot{a}_{\overline{4}|0.025}$ where $1 \le k \le 8$. The annual effective rate of interest is $1+i = (1.025)^4$. The accumulated value at time $t = 32$ is

$$\ddot{a}_{\overline{4}|0.025}(I\ddot{s})_{\overline{8}|i} = \ddot{a}_{\overline{4}|0.025}\tfrac{[s_{\overline{9}|i}-9](1+i)}{i} = 196.77.$$

On the other hand, we have

$$\frac{X}{(1.025)^4-1} = 196.77.$$

Solving this equation for X we find $X = 20.43$ $\blacksquare$

Practice Problems

Problem 28.1
(a) Find the total sum of all the payments in $(Ia)^{(12)}_{\overline{2}|}$.
(b) Find the total sum of all the payments in $(I^{(12)}a)^{(12)}_{\overline{2}|}$.

Problem 28.2
Find an expression for $(I^{(m)}a)^{(m)}_{\overline{\infty}|}$.

Problem 28.3
An annuity pays \$100 at the end of each month in the first year, \$200 at the end of each month in the second year, and continues to increase until it pays 1000 at the end of each month during the 10th year. Calculate the present value of the annuity at an annual effective interest rate of 6%.

Problem 28.4
A monthly annuity due pays \$1 at the beginning of the first month. Each subsequent payment increases by \$1. The last payment is made at the beginning of the 240th month. Calculate the accumulated value of the annuity at the end of the 240th month using:
(a) an interest rate of 6% compounded monthly;
(b) an annual effective interest rate of 6%.

Problem 28.5
A 20 year annuity pays 10 at the beginning of each quarter during the first year, 20 at the beginning of each quarter during the second year, etc with 200 being paid at the beginning of each quarter during the last year.
Calculate the present value of the annuity assuming an annual effective interest rate of 12%.

Problem 28.6
A 20 year annuity pays 10 at the beginning of each quarter during the first year, 20 at the beginning of each quarter during the second year, etc with 200 being paid at the beginning of each quarter during the last year.
Calculate the accumulated value of the annuity assuming a nominal interest rate of 6% compounded monthly.

Problem 28.7
Show that the present value of a perpetuity on which payments are 1 at the end of the 5th and 6th years, 2 at the end of the 7th and 8th years, 3 at the end of the 9th and 10th years, etc, is

$$\frac{\nu^4}{i-\nu d}.$$

Problem 28.8
A perpetuity has payments at the end of each four-year period. The first payment at the end of four years is 1. Each subsequent paymet is 5 more than the previous payment. It is known that $v^4 = 0.75$. Calculate the present value of this perpetuity.

Problem 28.9
A perpetuity provides payments every six months starting today. The first payment is 1 and each payment is 3% greater than the immediately preceding payment. Find the present value of the perpetuity if the effective rate of interest is 8% per annum.

Problem 28.10
Find the accumulated value at the end of ten years of an annuity in which payments are made at the beginning of each half-year for five years. The first payment is \$2,000, and each of the other payments is 98% of the previous payment. Interest is credited at 10% convertible quarterly.

Problem 28.11
Find the present value of payments of 5 now, 10 in 6 months, 15 in one year, 20 in 18 months, and so on for 6 years. The nominal discount rate is 12% convertible semiannually.

Problem 28.12
Marlen invests \$ X now in order to receive \$5 in 2 months, \$10 in 4 months, \$15 in 6 months, and so on. The payments continue for 10 years. The annual effective rate of interest is 8%. Determine X.

Problem 28.13
A 10-year annuity has the following schedule of payments: \$100 on each January 1, \$200 on each April 1, \$300 on each July 1, and \$400 on each October 1. Show that the present value of this annuity on January 1 just before the first payment is made is

$$1600\ddot{a}_{\overline{10|}}(I^{(4)}\ddot{a})^{(4)}_{\overline{1|}}.$$

Problem 28.14
Calculate the accumulated value at time 15 years of payments of 35 at the start of every quarter during the first year, 70 at the start of every quarter during the second year, 105 at the start of every quarter during the third year, and so on for 15 years. The nominal interest is 12% convertible quarterly.

Problem 28.15
A perpetuity pays 1000 immediately. The second payment is 97% of the first payment and is made at the end of the fourth year. Each subsequent payment is 97% of the previous payment and is paid four years after the previous payment. Calculate the present value of this annuity at an annual effective rate of 8%.

Problem 28.16
A 20 year annuity pays 10 at the end of the first quarter, 20 at the end of the second quarter, etc with each payment increasing by 10 until the last quarterly payment is made at the end of the 20th year. Calculate the present value of the annuity assuming a nominal interest rate of 6% compounded monthly.

Problem 28.17
A 20 year annuity pays 10 at the end of the first quarter, 20 at the end of the second quarter, etc with each payment increasing by 10 until the last quarterly payment is made at the end of the 20th year. Calculate the present value of the annuity assuming an annual effective interest rate of 12%.

Problem 28.18
Joan has won a lottery that pays 1000 per month in the first year, 1100 per month in the second year, 1200 per month in the third year, and so on. Payments are made at the end of each month for 10 years.
Using an effective interest rate of 3% per annum, calculate the present value of this prize. Round your answer to the nearest integer.

Problem 28.19
On his 65^{th} birthday Smith would like to purchase a ten-year annuity-immediate that pays 6000 per month for the first year, 5500 per month for the second year, 5000 per month for the third year, and so on. Using an annual effective interest rate of 5%, how much will Smith pay for such an annuity?

Problem 28.20
On Susan's 65^{th} birthday, she elects to receive her retirement benefit over 20 years at the rate of 2,000 at the end of each month. The monthly benefit increases by 4% each year.
Assuming an annual effective interest rate of 8.16%, and letting j denote the equivalent monthly effective interest rate, find the present value on Susan's 65^{th} birthday of her retirement benefit.

29 Continuous Varying Annuities

In this section we look at annuities in which payments are being made continuously at a varying rate.
Consider an annuity for n interest conversion periods in which payments are being made continuously at the rate $f(t)$ at exact moment t and the interest rate is variable with variable force of interest δ_t. Then $f(t)e^{-\int_0^t \delta_r dr}dt$ is the present value of the payment $f(t)dt$ made at exact moment t. Hence, the present value of this $n-$period continuous varying annuity is

$$PV = \int_0^n f(t)e^{-\int_0^t \delta_r dr}dt. \tag{29.1}$$

Example 29.1
Find an expression for the present value of a continuously increasing annuity with a term of n years if the force of interest is δ and if the rate of payment at time t is t^2 per annum.

Solution.
Using integration by parts process, we find

$$\begin{aligned}
\int_0^n t^2 e^{-\delta t}dt =& -\frac{t^2}{\delta}e^{-\delta t}\Big]_0^n + \frac{2}{\delta}\int_0^n te^{-\delta t}dt\\
=& -\frac{n^2}{\delta}e^{-\delta n} - \left[\frac{2t}{\delta^2}e^{-\delta t}\right]_0^n + \frac{2}{\delta^2}\int_0^n e^{-\delta t}dt\\
=& -\frac{n^2}{\delta}e^{-\delta n} - \frac{2n}{\delta^2}e^{-\delta n} - \left[\frac{2}{\delta^3}e^{-\delta t}\right]_0^n\\
=& -\frac{n^2}{\delta}e^{-\delta n} - \frac{2n}{\delta^2}e^{-\delta n} - \frac{2}{\delta^3}e^{-\delta n} + \frac{2}{\delta^3}\\
=& \frac{2}{\delta^3} - e^{-\delta n}\left[\frac{n^2}{\delta} + \frac{2n}{\delta^2} + \frac{2}{\delta^3}\right] \blacksquare
\end{aligned}$$

Under compound interest, i.e., $\delta_t = \ln(1+i)$, formula (29.1) becomes

$$PV = \int_0^n f(t)\nu^t dt.$$

Under compound interest and with $f(t) = t$ (an increasing annuity), the present value is

$$\begin{aligned}(\overline{I}\overline{a})_{\overline{n}|} =& \int_0^n t\nu^t dt\\ =& \left.\frac{t\nu^t}{\ln \nu}\right|_0^n - \int_0^n \frac{\nu^t}{\ln \nu}dt\\ =& \left.\frac{t\nu^t}{\ln \nu}\right|_0^n - \left.\frac{\nu^t}{(\ln \nu)^2}\right|_0^n\\ =& -\frac{n\nu^n}{\delta} - \frac{\nu^n}{\delta^2} + \frac{1}{\delta^2}\\ =& \frac{1-\nu^n}{\delta^2} - \frac{n\nu^n}{\delta}\\ =& \frac{\overline{a}_{\overline{n}|} - n\nu^n}{\delta}\end{aligned}$$

and the accumulated value at time n years is

$$(\overline{I}\overline{s})_{\overline{n}|} = (1+i)^n(\overline{I}\overline{a})_{\overline{n}|} = \frac{\overline{s}_{\overline{n}|} - n}{\delta}.$$

The above two formulas can be derived from the formulas of $(I^{(m)}a)^{(m)}_{\overline{n}|}$ and $(I^{(m)}s)^{(m)}_{\overline{n}|}$. Indeed,

$$(\overline{I}\overline{a})_{\overline{n}|} = \lim_{m\to\infty}(I^{(m)}a)^{(m)}_{\overline{n}|} = \lim_{m\to\infty}\frac{\ddot{a}^{(m)}_{\overline{n}|} - n\nu^n}{i^{(m)}} = \frac{\overline{a}_{\overline{n}|} - n\nu^n}{\delta}$$

and

$$(\overline{I}\overline{s})_{\overline{n}|} = \lim_{m\to\infty}(I^{(m)}s)^{(m)}_{\overline{n}|} = \lim_{m\to\infty}\frac{\ddot{s}^{(m)}_{\overline{n}|} - n}{i^{(m)}} = \frac{\overline{s}_{\overline{n}|} - n}{\delta}.$$

Example 29.2
Sam receives continuous payments at an annual rate of $8t + 5$ from time 0 to 10 years. The continuously compounded interest rate is 9%.
(a) Determine the present value at time 0.
(b) Determine the accumulated value at time 10 years.

Solution.
(a) The payment stream can be split into two parts so that the present value is

$$8(\overline{I}\overline{a})_{\overline{10}|} + 5\overline{a}_{\overline{10}|}.$$

Since

$$\begin{aligned} i =& e^{0.09} - 1 = 9.4174\% \\ \overline{a}_{\overline{10}|} =& \frac{1-(1.094174)^{-10}}{0.09} = 6.59370 \\ (\overline{I}\overline{a})_{\overline{10}|} =& \frac{6.59370 - 10(1.094174)^{-10}}{0.09} = 28.088592 \end{aligned}$$

we obtain

$$8(\overline{I}\overline{a})_{\overline{10}|} + 5\overline{a}_{\overline{10}|} = 8 \times 28.088592 + 5 \times 6.59370 = 257.68.$$

(b) The accumulated value at time 10 years is

$$257.68 \times (1.094174)^{10} = 633.78 \blacksquare$$

Example 29.3
If $\delta = 0.06$, calculate the present value of 10 year continuous annuity payable at a rate of t at time t.

Solution.
The present value is

$$(\overline{I}\overline{a})_{\overline{10}|} = \frac{1-\nu^{10}}{\delta^2} - \frac{10\nu^{10}}{\delta} = \frac{1-e^{-10\times 0.06}}{0.06^2} - \frac{10e^{-10\times 0.06}}{0.06} = 33.86 \blacksquare$$

For a continuous payable continuously increasing perpetuity (where $f(t) = t$), the present value at time 0 is

$$(\overline{I}\overline{a})_{\overline{\infty}|} = \lim_{n\to\infty} \frac{\overline{a}_{\overline{n}|} - n\nu^n}{\delta} = \lim_{n\to\infty} \frac{\frac{1-(1+i)^{-n}}{\delta} - n(1+i)^{-n}}{\delta} = \frac{1}{\delta^2}.$$

Example 29.4
Determine the present value of a payment stream that pays a rate of $5t$ at time t. The payments start at time 0 and they continue indefinitely. The annual effective interest rate is 7%.

Solution.
The present value is

$$5(\overline{I}\overline{a})_{\overline{\infty}|} = \frac{5}{[\ln(1.07)]^2} = 1,092.25 \blacksquare$$

We conclude this section by considering the case of a continuously decreasing continuously payable stream in which a continuous payment is received from time 0 to time n years. The rate of payment

at time t is $f(t) = n - t$, and the force of interest is constant at δ.
The present value is

$$\begin{aligned}(\overline{D}\overline{a})_{\overline{n}|} =& n\overline{a}_{\overline{n}|} - (\overline{I}\overline{a})_{\overline{n}|}\\ =& n\frac{1-\nu^n}{\delta} - \frac{\overline{a}_{\overline{n}|} - n\nu^n}{\delta}\\ =& \frac{n - \overline{a}_{\overline{n}|}}{\delta}\end{aligned}$$

Example 29.5
Otto receives a payment at an annual rate of $10 - t$ from time 0 to time 10 years. The force of interest is 6%. Determine the present value of these payments at time 0.

Solution.
Since

$$i = e^{0.06} - 1 = 6.184\%$$

$$\overline{a}_{\overline{10}|} = \frac{1 - (1.06184)^{-10}}{0.06} = 7.5201$$

the present value is then

$$(\overline{D}\overline{a})_{\overline{10}|} = \frac{10 - 7.5201}{0.06} = 41.33 \blacksquare$$

Example 29.6
Using the information from the previous example, determine the accumulated value at time 10 years of the payments received by Otto.

Solution.
The accumulated value at time 10 years is

$$(1.06184)^{10} \times (\overline{D}\overline{a})_{\overline{10}|} = (1.06184)^{10} \times 41.33 = 75.31 \blacksquare$$

Practice Problems

Problem 29.1
Calculate the present value at an annual effective interest rate of 6% of a 10 year continuous annuity which pays at the rate of t^2 per period at exact moment t.

Problem 29.2
Calculate the accumulated value at a constant force of interest of 5% of a 20 year continuous annuity which pays at the rate of $\frac{t+1}{2}$ per period at exact moment t.

Problem 29.3
Evaluate $(\overline{I}\overline{a})_{\overline{\infty|}}$ if $\delta = 0.08$.

Problem 29.4
Payments under a continuous perpetuity are made at the periodic rate of $(1+k)^t$ at time t. The annual effective rate of interest is i, where $0 < k < i$. Find an expression for the present value of the perpetuity.

Problem 29.5
A perpetuity is payable continuously at the annual rate of $1 + t^2$ at time t. If $\delta = 0.05$, find the present value of the perpetuity.

Problem 29.6
A one-year deferred continuous varying annuity is payable for 13 years. The rate of payment at time t is $t^2 - 1$ per annum, and the force of interest at time t is $(1+t)^{-1}$. Find the present value of the annuity.

Problem 29.7
If $\overline{a}_{\overline{n|}} = a$ and $\overline{a}_{\overline{2n|}} = b$, express $(\overline{I}\overline{a})_{\overline{n|}}$ in terms of a and b.

Problem 29.8
A 10 year continuous annuity pays at a rate of $t^2 + 2t + 1$ at time t. Calculate the present value of this annuity if $\delta_t = (1+t)^{-1}$.

Problem 29.9
A 30 year continuous annuity pays at a rate of t at time t. If $\delta = 0.10$, calculate the current value of the annuity after ten years.

Problem 29.10
A 10 year continuous annuity pays at a rate of t at time t. If $\nu^t = .94^t$, calculate the accumulated value of the annuity after ten years.

Problem 29.11
Nancy receives a payment stream from time 2 to time 7 years that pays an annual rate of $2t-3$ at time t. The force of interest is constant at 6% over the period. Calculate the accumulated value at time 7 years.

Problem 29.12
Find the ratio of the total payments made under $(\overline{I}\overline{a})_{\overline{10}|}$ during the second half of the term of the annuity to those made during the first half.

Problem 29.13 ‡
Payments are made to an account at a continuous rate of $(8k+tk)$, where $0 \le t \le 10$. Interest is credited at a force of interest $\delta_t = \frac{1}{8+t}$. After 10 years, the account is worth 20,000. Calculate k.

Problem 29.14
Show that $(\overline{I}\overline{a})_{\overline{n}|} < (I^{(m)}a)^{(m)}_{\overline{n}|} < (Ia)^{(m)}_{\overline{n}|}$ whenever $n\delta < 1$. Hint: Use Problem 25.11.

Problem 29.15
An investment fund is started with an initial deposit of 1 at time 0. New deposits are made continuously at the annual rate $1+t$ at time t over the next n years. The force of interest at time t is given by $\delta_t = (1+t)^{-1}$. Find the accumulated value in the fund at the end of n years.

Problem 29.16
Money was deposited continuously at the rate of 5,000 per annum into a fund for 7 years. If the force of interest is 0.05 from $t=0$ to $t=10$, what will the fund balance be after 10 years?

Problem 29.17
A 5−year annuity makes payments continuously at a rate $f(t) = e^{\left(\frac{t^2}{24}+t\right)}$. The force of interest varies continuously at a rate of $\delta_t = \frac{t}{12}$. Find the present value of this annuity.

Rate of Return of an Investment

In this chapter, we present a number of important results for using the theory of interest in more complex "real-world" contexts than considered in the earlier chapters.
In this chapter, the concept of yield rates is introduced: A **yield rate** (also known as the **internal rate of return**) is that interest rate at which the present value of all returns from the investment is equal to the present value of all contributions into the investment. Thus, yield rates are solutions to the equation $NPV(i) = 0$ where NPV stands for the **net present value** and is the difference between the present value of all returns and that of all contributions. Techniques for measuring the internal rate of return will be discussed.
Yield rates are used as an index to see how favorable or unfavorable a particular transaction is. When lower, yield rates favor the borrower and when higher they favor the lender.
We point out here, as always the case thus far, that the effect of taxes will be ignored throughout the discussions.

30 Discounted Cash Flow Technique

In this section we consider the question of profitability of investment projects that involve cash flows. More specifically, we look at the present values of projects consisting of cash flows. Such projects can be considered as a generalization of annuities with any pattern of payments and/or withdrawals compared to annuities previously discussed where the annuities consisted of regular series of payments. The process of finding the present value of the type of projects consisting of a stream of cash flows will be referred to as the **discounted cash flow technique**, abbreviated by DCF. We examine two DCF measures for assessing a project: **net present value** and **internal rate of return.**
Consider an investment project with contributions (outflow) $C_0, C_1, \cdots, C_n$ at time $t_0 < t_1 < t_2 < \cdots < t_n$ and returns (inflow) $R_0, R_1, \cdots, R_n$ at the same point of time. Let $c_t = R_t - C_t$ be the net change (or **net cash flow**) in the account at time t which can be positive or negative.
From the vantage point of the lender, if $c_t > 0$, then there is a net cash deposit into the investment at time t. If $c_t < 0$ then there is a net cash withdrawal from the investment at time t. Thus, if at time 1 a deposit of \$1,000 is made and at the same time a withdrawal of \$2,000 is made then $c_1 = 1000 - 2000 = -1,000$ so there is a net cash withdrawal of \$1,000 from the investment at that time.
We have adopted the vantage point of the lender. From the vantage point of the borrower the signs in the above discussion are switched. The following example illustrates these definitions.

Example 30.1
In order to develop a new product and place it in the market for sale, a company has to invest \$80,000 at the beginning of the year and then \$10,000 for each of the next three years. The product is made available for sale in the fourth year. For that, a contribution of \$20,000 must be made in the fourth year. The company incurs maintenance expenses of \$2000 in each of the next five years. The project is expected to provide an investment return of \$12,000 at the end of the fourth year, \$30,000 at the end of the fifth year, \$40,000 at the end of the sixth year, \$35,000 at the end of the seventh year, \$25,000 at the end of the eigth year, \$15,000 at the end of the ninth year and \$8,000 at the end of the tenth year. After the tenth year, the product is withdrawn from the market. Create a chart to describe the cash flows of this project.

Solution.
Table 30.1 describes the cash flows of this project ■

Year	Contributions	Returns	c_t
0	80,000	0	$-80,000$
1	10,000	0	$-10,000$
2	10,000	0	$-10,000$
3	10,000	0	$-10,000$
4	20,000	12,000	$-8,000$
5	2,000	30,000	28,000
6	2,000	40,000	38,000
7	2,000	35,000	33,000
8	2,000	25,000	23,000
9	2,000	15,000	13,000
10	0	8,000	8,000

Table 30.1

Now, suppose that the rate of interest per period is i. This interest rate is sometimes referred to as the **required return of the investment** or the **cost of capital**. Using the discounted cash flow technique, the **net present value** at rate i of the investment is defined by

$$NPV(i) = \sum_{k=0}^{n} \nu^{t_k} c_{t_k}, \quad \nu = (1+i)^{-1}.$$

Thus, the net present value of a series of cash flows is the present value of cash inflows minus the present value of cash outflows. Expressed in another way, NPV is the sum of the present values of the net cash flows over n periods.
The value of $NPV(i)$ can be positive, negative, or zero depending on how large or small the value of i is.

Example 30.2
Find the net present value of the investment discussed in the previous example.

Solution.
The net present value is

$$\begin{aligned} NPV(i) = & -80,000 - 10,000\nu - 10,000\nu^2 - 10,000\nu^3 - 8,000\nu^4 + 28,000\nu^5 \\ & + 38,000\nu 6 + 33,000\nu^7 + 23,000\nu^8 + 13,000\nu^9 + 8,000\nu^{10} \end{aligned}$$

Notice that $NPV(0.03) = 1488.04 > 0$, $NPV(0.032180) = 0$, and $NPV(0.04) = -5122.13 < 0$ ■

The value of i for which $NPV(i) = 0$ is called the **yield rate** (or **internal rate of return**);

that is,
the yield rate is the rate of interest at which the present value of contributions from the investment is equal to the present value of returns into the investment.
Stated differently, the yield rate is the rate that causes investment to break even.
According to the eqution $NPV(i) = 0$, yield rates are the same from either the borrower's or lender's perspective. They are totally determined by the cash flows defined in the transaction and their timing.
Yield rates are often used to measure how favorable or unfavorable a transaction might be. From a lender's perspective, a higher yield rate makes a transaction more favorable. From a borrower's perspective, a lower yield rate makes a transaction more favorable.
Based on financial factors alone, projects with positive NPV are considered acceptable. Projects with $NPV(i) < 0$ should be rejected. When $NPV(i) = 0$, the investment would neither gain nor lose value; We should be indifferent in the decision whether to accept or reject the project.

Example 30.3
Determine the net present value of the investment project with the following cash flow, at a cost of capital of 4.8%.

Time	0	1	2	3	4	5
Flow	−1000	70	70	70	70	1070

Solution.
The answer is $NPV(i) = -1000 + 70a_{\overline{5}|} + 1000\nu^5 = 95.78$ ■

Example 30.4
Assume that cash flows from the construction and sale of an office building is as follows.

Year	0	1	2
Amount	−150000	−100000	300000

Given a 7% cost of capital, create a present value worksheet and show the net present value, NPV.

Solution.

t	v^t	c_t	PV
0	1.0	−150000	−150000
1	$\frac{1}{1.07} = 0.935$	−100000	−93500
2	$\frac{1}{1.07^2} = 0.873$	300000	261900
		NPV=	18400 ■

Finding the yield rate may require the use of various approximation methods, since the equations that have to be solved may be polynomials of high degree.

Example 30.5
An investment project has the following cash flows:

Year	Contributions	Returns
0	100	0
1	200	0
2	10	60
3	10	80
4	10	100
5	5	120
6	0	60

(a) Calculate the net present value at 15%.
(b) Calculate the internal rate of return (i.e. the yield rate) on this investment.

Solution.
(a) $NPV(0.15) = -100 - 200(1.15)^{-1} + 50(1.15)^{-2} + 70(1.15)^{-3} + 90(1.15)^{-4} + 115(1.15)^{-5} + 60(1.15)^{-6} = -\$55.51$.
(b) We must solve the equation $-100 - 200\nu + 50\nu^2 + 70\nu^3 + 90\nu^4 + 115\nu^5 + 60\nu^6 = 0$. Using linear interpolation with the points $(7\%, 4.53)$ and $(8\%, -4.52)$ one finds

$$i \approx 0.07 - 4.53 \times \frac{0.08 - 0.07}{-4.52 - 4.53} = 7.50\% \blacksquare$$

Yield rate needs not be unique as illustrated in the following example.

Example 30.6
In exchange for receiving \$230 at the end of one year, an investor pays \$100 immediately and pays \$132 at the end of two years. Find the yield rate.

Solution.
The net present value is $NPV(i) = -100 - 132\nu^2 + 230\nu$. To find the IRR we must solve $132\nu^2 - 230\nu + 100 = 0$. Solving this equation by the quadratic formula, we find $i = 10\%$ or $i = 20\%$ $\blacksquare$

Consider the situation where a bank makes a loan to an individual. In this case the bank acts as a lender or investor, and the individual acts as the borrower. Payments that are received by the investor (cash inflows) are taken to be positive cash flows. Payments that are made by the investor (cash outflows) are considered to be negative. For the borrower, the cash flows will be of the opposite sign.

Example 30.7
A bank lends Diane \$8,000 now. She repays \$600 at the end of each quarter for 5 years. Find the annual effective rate of return.

Solution.
Let j be the effective IRR per quarter. Then j satisfies the equation $-8000 + 600a_{\overline{20}|j} = 0$ or

$$3(1 - (1 + j)^{-20}) - 40j = 0.$$

Using linear interpolation with the points $(4\%, 0.030839)$ and $(4.5\%, -0.04392)$ we find

$$j \approx 0.04 - 0.030839 \times \frac{0.045 - 0.04}{-0.04392 - 0.030839} = 0.042.$$

Thus, the annual rate of return is

$$i = 1.042^4 - 1 = 17.89\% \blacksquare$$

Yield rates need not be positive. If a yield rate is 0, then the investor(lender) received no return on investment. If a yield rate is negative, then the investor(lender) lost money on the investment. We will assume that such a negative yield rate satisfies $-1 < i < 0$. A yield rate $i < -1$ implies full loss of the investment.

Example 30.8
Find the yield rates of Example 30.1

Solution.
Using a scientific calculator one finds the two real solutions $i = -1.714964$ and $i = 0.032180$. Since, $i > -1$, the only yield rate is $i = 3.218\% \blacksquare$

Practice Problems

Problem 30.1
Assuming a nominal rate of interest, convertible quarterly, of 12%, find the net present value of a project that requires an investment of \$100,000 now, and returns \$16,000 at the end of each of years 4 through 10.

Problem 30.2
The internal rate of return for an investment with contributions of \$3,000 at time 0 and \$1,000 at time 1 and returns of \$2,000 at time 1 and \$4,000 at time 2 can be expressed as $\frac{1}{n}$. Find n.

Problem 30.3
An investor enters into an agreement to contribute \$7,000 immediately and \$1,000 at the end of two years in exchange for the receipt of \$4,000 at the end of one year and \$5,500 at the end of three years. Find $NPV(0.10)$.

Problem 30.4
An investment project has the following cash flows:

Year	Contributions	Returns
0	100,000	0
1	5,000	0
2	4,000	10,000
3	2,000	10,000
4	0	20,000
5	0	40,000
6	0	60,000
7	0	80,000

(a) Calculate the net present value at 15%.
(b) Calculate the internal rate of return on this investment.

Problem 30.5
You are the President of XYZ Manufacturing Company. You are considering building a new factory. The factory will require an investment of 100,000 immediately. It will also require an additional investment of 15,000 at the beginning of year 2 to initiate production. Finally, maintenance costs for the factory will be 5,000 per year at the beginning of years 3 through 6.
The factory is expected to generate profits of 10,000 at the end of year one, 15,000 at the end of year two, 20,000 at the end of year 3, and 30,000 at the end of years 4 through 6.
Calculate the internal rate of return on the potential factory.

Problem 30.6
An investment project has the following cash flows:

Year	Contributions	Returns
0	100,000	0
1	0	10,000
2	0	20,000
3	0	30,000
4	0	20,000
5	0	10,000

Calculate the internal rate of return on this investment.

Problem 30.7
What is the internal rate of return of a project that requires an investment of \$1,000,000 now, and returns \$150,000 at the end of each of years 1 through 15?

Problem 30.8
Assuming an annual effective discount rate of 6%, what is the net present value of a project that requires an investment of \$75,000 now, and returns $2,000(13-t)$ at times $t = 1, 2, \cdots, 12$ (i.e, \$24,000 at time 1, \$22,000 at time 2, \$20,000 at time 3, etc.)?

Problem 30.9
What is the internal rate of return on a project that requires a \$10,000 investment now, and returns \$4,000 four years from now and \$12,000 eight years from now?

Problem 30.10
A project has the following out-flows and in-flows:
out-flows: 12,000 at $t = 0$, 6,000 at $t = 3$, 9,000 at $t = 6$ and 12,000 at $t = 9$
in-flows: 3,000 at $t = 2$ through $t = 14$ plus 12,000 at $t = 15$.
If your desired internal rate of return was 8%, would you accept this project?

Problem 30.11
A company pays \$120,000 to purchase a property. The company pays \$3,000 at the end of each of the next 6 months to renovate the property. At the end of the eigth month, the company sells the property for \$150,000. Find the net present value of this project for the company at an annual effective rate of 8%.

Problem 30.12
What is the project's monthly internal rate of return of the previous problem?

Problem 30.13
A project requires an initial investment of $50,000. The project will generate net cash flows of $15,000 at the end of the first year, $40,000 at the end of the second year, and $10,000 at the end of the third year. The project's cost of capital is 13%. According to the DCF technique, should you invest in the project?

Problem 30.14
A ten-year investment project requires an initial investment of $100,000 at inception and maintenance expenses at the beginning of each year. The maintenance expense for the first year is $3,000, and is anticipated to increase 6% each year thereafter. Projected annual returns from the project are $30,000 at the end of the first year decreasing 4% per year thereafter. Find $c_6 = R_6 - C_6$.

Problem 30.15
Determine the net present value for a project that costs $104,000 and would yield after−tax cash flows of $16,000 the first year, $18,000 the second year, $21,000 the third year, $23,000 the fourth year, $27,000 the fifth year, and $33,000 the sixth year. Your firm's cost of capital is 12%.

Problem 30.16
Determine the net present value for a project that costs $253,494.00 and is expected to yield after−tax cash flows of $29,000 per year for the first ten years, $37,000 per year for the next ten years, and $50,000 per year for the following ten years. Your firm's cost of capital is 12%.

Problem 30.17
An investment project has the following cash flows:

Year	Contributions	Returns
0	10,000	0
1	5,000	0
2	1,000	0
3	1,000	0
4	1,000	0
5	1,000	0
6	1,000	8,000
7	1,000	9,000
8	1,000	10,000
9	1,000	11,000
10	0	12,000

Find the yield rate of this investment project.

Problem 30.18
Suppose a sum of money could be invested in project A which pays a 10% effective rate for six years, or be invested in project B which pays an 8% effective rate for 12 years.
(a) Which option we choose as an investor?
(b) Find the effective rate of interest necessary for the six years after project A ends so that investing in project B would be equivalent.

Problem 30.19 ‡
Project P requires an investment of 4000 at time 0. The investment pays 2000 at time 1 and 4000 at time 2.
Project Q requires an investment of X at time 2. The investment pays 2000 at time 0 and 4000 at time 1.
The net present values of the two projects are equal at an interest rate of 10%. Calculate X.

Problem 30.20
X corporation must decide whether to introduce a new product line. The new product will have startup costs, operational costs, and incoming cash flows over six years. This project will have an immediate ($t = 0$) cash outflow of \$100,000 (which might include machinery, and employee training costs). Other cash outflows for years 1-6 are expected to be \$5,000 per year. Cash inflows are expected to be \$30,000 per year for years 1-6. All cash flows are after-tax, and there are no cash flows expected after year 6. The required rate of return is 10%. Determine the net present value at this rate.

Problem 30.21
An investment account is established on which it is estimated that 8% can be earned over the next 20 years. If the interest each year is subject to income tax at a 25% tax rate, find the percentage reduction in the accumulated interest at the end of 20 years.

31 Uniqueness of IRR

As pointed out in Example 30.4, an internal rate of return may not be unique. In this section, we look at conditions under which the equation $\sum_{i=0}^{n} c_{t_i}\nu^{t_i} = 0$ has exactly one solution. For the ease of the analysis we choose time to be equally spaced so that $NPV(i) = \sum_{t=0}^{n} c_t\nu^t = 0$, that is, the net present value function is a polynomial of degree n in ν. By the Fundamental Theorem of Algebra, $NPV(i) = 0$ has n roots, counting repeated roots and complex roots.

Example 31.1
Consider an investment with $C_0 = 1, C_1 = 0, C_2 = 1.36, R_0 = 0, R_1 = 2.3, R_2 = 0$. Find the internal rate of return.

Solution.
We have $c_0 = -1, c_1 = 2.3, c_2 = -1.36$. The internal rate of return is a solution to the equation $-1 + 2.3\nu - 1.36\nu^2 = 0$. Solving this equation using the quadratic formula we find $\nu = \frac{2.3 \pm i\sqrt{0.0111}}{2.72}$ so no real root for i ■

Example 31.2
Consider an investment with $C_0 = 1, C_1 = 0, C_2 = 1.32, R_0 = 0, R_1 = 2.3, R_2 = 0$. Find the internal rate of return.

Solution.
We have $c_0 = -1, c_1 = 2.3, c_2 = -1.32$. Thus, the equation of value at time $t = 0$ is $-1 + 2.3\nu - 1.32\nu^2 = 0$. Solving this equation using the quadratic formula we find $\nu = \frac{-2.3 \pm 0.1}{-2.64}$ so either $\nu = 0.833$ or $\nu = 0.91$. Hence, either $i = 20\%$ or $i = 10\%$ so two internal rates of return ■

Example 31.3
Consider an investment with $C_0 = 1, C_1 = 0, C_2 = 1.2825, R_0 = 0, R_1 = 2.3, R_2 = 0$. Find the internal rate of return.

Solution.
We have $c_0 = -1, c_1 = 2.3, c_2 = -1.2825$. Thus, the equation of value at time $t = 0$ is $-1 + 2.3\nu - 1.2825\nu^2 = 0$. Solving this equation we find $\nu = \frac{-2.3 \pm 0.4}{-2.565}$ so either $\nu = 0.74$ or $\nu = 1.053$. Hence, either $i = 35\%$ or $i = -5\%$ ■

What we would like to do next is to formulate conditions on the c_t that guarantee a unique yield rate $i > -1$. A first set of conditions is given in the following theorem.

Theorem 31.1
Let k be an integer such that $0 < k < n$. Suppose that either
(i) $c_t \leq 0$ for $0 \leq t \leq k$ and $c_t \geq 0$ for $k+1 \leq t \leq n$; or
(ii) $c_t \geq 0$ for $0 \leq t \leq k$ and $c_t \leq 0$ for $k+1 \leq t \leq n$.
Then there is a unique interest rate $i > -1$ such that $NPV(i) = 0$.

This theorem states that a unique internal rate of return exists for a transaction in which the net payments are all of one sign for the first portion of the transaction and then have the opposite sign for the remainder of the transaction.
To prove the theorem, we require the concept of Descartes' Rule of signs which is a technique for determining the number of positive or negative roots of a polynomial:
If the terms of a single-variable polynomial with real coefficients and a nonzero constant term are ordered by descending variable exponent, then the number of positive roots counting multiplicity of a polynomial $f(x)$ is either equal to the number of sign changes between adjacent nonzero coefficients, or less than it by a multiple of 2. The number of negative zeros of a polynomial $f(x)$ counting multiplicity is equal to the number of positive zeros of $f(-x)$, so it is the number of changes of signs of $f(-x)$ or that number decreased by an even integer.
Complex roots always come in pairs. That's why the number of positive or number of negative roots must decrease by two.

Example 31.4
Use Descartes' rule of sign to find the possible number of positive roots and negative roots of the polynomial $f(x) = x^2 + x + 1$.

Solution.
The polynomial $f(x)$ has no sign change between adjacent coefficients. Thus, $f(x)$ has no positive roots. The polynomial $f(-x) = x^2 - x + 1$ has two sign changes between adjacent coefficients so that $f(x)$ has either two negative roots or zero negative roots. Thus, the two possibilities are either two negative roots or two complex roots ■

Example 31.5
Use Descartes' rule of sign to find the possible number of positive roots and negative roots of the polynomial $f(x) = x^3 + x^2 - x - 1$.

Solution.
The polynomial $f(x)$ has one sign change between the second and third terms. Therefore it has at most one positive root. On the other hand, $f(-x) = -x^3 + x^2 + x - 1$ has two sign changes, so $f(x)$ has 2 or 0 negative roots. Since complex roots occur in conjugate, the two possibilities are either one positive root and two negative roots or one positive root and two complex roots. Thus, $f(x)$

has exactly one positive root ■

Proof of Theorem 31.1.
According to either (i) and (ii), the coefficients of $NPV(i)$ changes signs once. So by Descartes' rule of signs, there will be at most one positive real root. If such a root exists, then $\nu > 0$ implies $i > -1$. For such values of i we define the function

$$f(i) = (1+i)^k NPV(i) = \sum_{t=0}^{k} c_t(1+i)^{k-t} + \sum_{t=k+1}^{n} c_t(1+i)^{k-t}.$$

Taking the first derivative we find

$$f'(i) = \sum_{t=0}^{k-1} c_t(k-t)(1+i)^{k-t-1} + \sum_{t=k+1}^{n} c_t(k-t)(1+i)^{k-t-1}.$$

If (i) is satisfied then $c_t(k-t) \le 0$ for $0 \le t \le n$. Hence, $f'(i) \le 0$ and the function $f(i)$ is decreasing. If (ii) is satisfied then $f'(i) \ge 0$ and $f(i)$ is increasing.
In case (i), the function $f(i)$ is decreasing. Moreover, as $i \to -1^+$ the first sum in $f(i)$ approaches c_k whereas the second sum approaches ∞. Hence, $\lim_{i\to -1^+} f(i) = \infty$. Similarly, as i approaches $+\infty$ the first sum in $f(i)$ approaches $-\infty$ whereas the second sum approaches 0. Hence, $\lim_{i\to\infty} f(i) = -\infty$. Since f is decreasing, there is a unique $i > -1$ such that $NPV(i) = 0$. A similar argument holds for case (ii) ■

Example 31.6
A project requires an initial investment of \$10,000 and it produces net cash flows of \$10,000 one year from now and \$2,000 two years from now. Show that there is a unique internal rate of return.

Solution.
The net present value is $NPV(i) = -10,000 + 10,000\nu + 2,000\nu^2$. The result follows from Theorem 31.1(i) with $k = 1$ ■
The following theorem establishes uniqueness of yield rates under a broader set of conditions than the one given in the previous theorem.

Theorem 31.2

Let $i > -1$ be a solution to $NPV(i) = 0$. Suppose that

$$\begin{aligned}
B_0 =& c_0 > 0 \\
B_1 =& c_0(1+i) + c_1 > 0 \\
B_2 =& c_0(1+i)^2 + c_1(1+i) + c_2 > 0 \\
&\vdots \\
B_{n-1} =& c_0(1+i)^{n-1} + c_1(1+i)^{n-2} + \cdots + c_{n-1} > 0
\end{aligned}$$

Then
(i) $B_n = c_0(1+i)^n + c_1(1+i)^{n-1} + \cdots + c_n = 0$
(ii) i is unique.

Proof.
(i) We are given that $\sum_{t=0}^{n} c_t\nu^t = 0$. This is an nth degree polynomial in ν and could be written as an nth degree polynomial in i by simply multiplying both sides by $(1+i)^n$ to obtain

$$c_0(1+i)^n + c_1(1+i)^{n-1} + \cdots + c_n = 0.$$

Thus, the assumption of (i) is established.
(ii) Suppose that j satisfies $NPV(j) = 0$ and $j > i$. Then we have

$$\begin{aligned}
B'_0 =& c_0 = B_0 > 0 \\
B'_1 =& B'_0(1+j) + c_1 = c_0(1+j) + c_1 > B_1 > 0 \\
B'_2 =& B'_1(1+j) + c_2 = c_0(1+j)^2 + c_1(1+j) + c_2 > B_2 > 0 \\
&\vdots \\
B'_{n-1} =& c_0(1+j)^{n-1} + c_1(1+j)^{n-2} + \cdots + c_{n-1} > B_{n-1} > 0 \\
B'_n =& B'_{n-1}(1+j) + c_n = c_0(1+j)^n + c_1(1+j)^{n-1} + \cdots + c_n > B_n = 0
\end{aligned}$$

The last inequality shows that j is not a yield rate, a contradiction. A similar argument holds for $-1 < j < i$. Hence, we conclude that $j = i$ ■

Remark 31.1
Note that B_t is the outstanding balance at time t. According to the theorem above, if the outstanding balance is positive throughout the period of investment, then the yield rate will be unique. Also, note that $c_0 > 0$ and $c_n = -B_{n-1}(1+i) < 0$, but that c_t for $t = 1, 2, \cdots, n-1$ may be either positive, negative, or zero.

Example 31.7
Show that we cannot guarantee uniqueness of the yield rate if c_0 and c_n have the same sign.

Solution.
The conclusion of uniqueness depends on the outstanding balance being positive at all time during the investment. If $c_0 > 0$ and $c_n > 0$ then the outstanding balance must be negative prior to $t = n$. The negative outstanding balance implies no guarantee of uniqueness. Similarly, if $c_0 < 0$ and $c_n < 0$, the outstanding balance at the start of the investment is negative so that no guarantee of uniqueness of yield rate ■

The discussion of this section focused on the existence of yield rates. However, it is possible that no yield rate exists or all yield rates are imaginary (See Example 31.1).

Example 31.8
An investor is able to borrow \$1,000 at 8% effective for one year and immediately invest the \$1,000 at 12% effective for the same year. Find the investor's yield rate on this transaction.

Solution.
The profit at the end of the year is \$40, but there is no yield rate, since the net investment is zero ■

Practice Problems

Problem 31.1
Use Descartes' rule of signs to determine the maximum number of real zeroes of the polynomial $f(x) = x^5 - x^4 + 3x^3 + 9x^2 - x + 5$.

Problem 31.2
Consider the following transaction: Payments of $100 now and $108.15 two years from now are equivalent to a payment of $208 one year from now. Find the absolute value of the difference of the two rates that result from the equation of value at time $t = 0$.

Problem 31.3
An investor pays $100 immediately and $X at the end of two years in exchange for $200 at the end of one year. Find the range of X such that two yield rates exist which are equal in absolute value but opposite in sign.

Problem 31.4
What is the yield rate on a transaction in which an investor makes payments of $100 immediately and $101 at the end of two years, in exchange for a payment of $200 at the end of one year?

Problem 31.5 ‡
A company deposits 1,000 at the beginning of the first year and 150 at the beginning of each subsequent year into perpetuity.
In return the company receives payments at the end of each year forever. The first payment is 100. Each subsequent payment increases by 5%.
Calculate the company's yield rate for this transaction.

Problem 31.6
An investor enters into an agreement to contribute $7,000 immediately and $1,000 at the end of two years in exchange for the receipt of $4,000 at the end of one year and $5,500 at the end of three years. What is the maximum number of possible yield rates using Descartes' rule of signs?

Problem 31.7
An investment costs $85 and it will pay $100 in 5 years. Determine the internal rate of return for this investment.

Problem 31.8
A project requires an initial investment of $50,000. The project will generate net cash flows of $15,000 at the end of the first year, $40,000 at the end of the second year, and $10,000 at the end of the third year. Show that an internal rate of return exists and is unique.

Problem 31.9
Suppose payments of 100 now and 109.20 two years from now are to be made in return for receiving 209 one year from now. Determine the internal rate of return.

Problem 31.10
What is the internal rate of return (in terms of an effective annual interest rate) on a project that requires a \$1,000 investment now, and provides returns to the investor of \$500 one year from now and \$800 two years from now?

Problem 31.11
Suppose a project requires you to invest \$10,000 now and \$20,000 one year from now. The project returns \$29,000 six months from now. Find all the yield rates (internal rates of return) of this project. Express these yield rates as annual effective rates.

Problem 31.12
For the previous problem, find the range of annual interest rates which will produce a net present value greater than zero.

Problem 31.13
Two growing perpetuities, each with annual payments, have the same yield rate (internal rate of return). The first perpetuity has an initial payment of \$50 one year from now, and each subsequent annual payment increases by \$5. The present value of this first perpetuity is \$2,000. The second perpetuity (a perpetuity-due) has an initial payment of \$100 now, and each subsequent annual payment increases by 3%. Find the present value, now, of the second perpetuity.

32 Interest Reinvested at a Different Rate

In this section we consider transactions where interest may be reinvested at a rate which may or may not be equal to the original investment rate. We analyze two situations in which reinvestment rates are directly taken into consideration.
First, consider an investment of 1 for n periods at rate i where the interest is reinvested at rate j. The time diagram of this situation is shown in Figure 32.1.

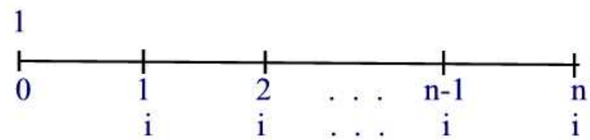


Figure 32.1

The accumulated value at the end of the n periods is equal to the principal plus the accumulated value of the annuity-immediate with periodic payments of i at the end of each period and periodic rate j. That is,

$$AV = 1 + is_{\overline{n}|j}.$$

Note that in the case $i = j$, the formula reduces to the familiar formula $AV = (1+i)^n$. That is, for compound interest, the reinvestment rate is equal to the original rate.

Example 32.1
100 is invested in a Fund A now. Fund A will pay interest at 10% each year. When the interest is paid in Fund A, it will be immediately removed and invested in Fund B paying 8 %. Calculate the sum of the amounts in Fund A and Fund B after 10 years.

Solution.
At the end of 10 years, Fund A will just have the original principal: \$100. Fund B will have the accumulated value of the interest payments at the rate 8%:

$$10s_{\overline{10}|0.08} = 10\frac{(1.08^{10} - 1)}{0.08} = 144.87.$$

So the total amount of money in both funds is about \$244.87 ■

Example 32.2
Brown deposits 5000 in a 5-year investment that pays interest quarterly at 8%, compounded quarterly. Upon receipt of each interest payment, Brown reinvests the interest in an account that earns 6%, compounded quarterly.
Determine Brown's yield rate over the 5 year period, as a nominal interest rate compounded quarterly.

Solution.
The equation of value after 20 quarters is

$$5000\left(1+\frac{i^{(4)}}{4}\right)^{20}=5000+5000(0.08)s_{\overline{20}|0.015}.$$

Solving this equation we find

$$i^{(4)}=4\left[1+0.08s_{\overline{20}|0.015}\right]^{\frac{1}{20}}-1=7.68\% \blacksquare$$

Second, consider an investment of 1 at the end of each period for n periods at rate i where the interest is reinvested at rate j. The time diagram of this situation is shown in Figure 32.2.

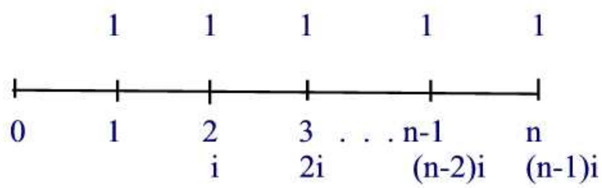


Figure 32.2

The accumulated value at the end of the n periods is the sum of the annuity payments and the accumulated value of the interest,i.e.

$$AV=n+i(Is)_{\overline{n-1}|j}=n+i\left[\frac{s_{\overline{n}|j}-n}{j}\right].$$

Note that the above formula simplifies to the familiar result $AV=s_{\overline{n}|}$ when $j=i$.

Example 32.3
Payments of $1,000 are invested at the end of each year for 10 years. The payments earn interest at 7% effective, and the interest is reinvested at 5% effective. Find the
(a) amount in the fund at the end of 10 years,
(b) purchase price an investor(a buyer) must pay for a yield rate (to the seller) of 8% effective.

Solution.
(a) The amount in the fund at the end of 10 years is

$$\begin{aligned}1000\left(10+0.07\left[\frac{s_{\overline{10}|0.05}-10}{0.05}\right]\right)=&1000\left(10+0.07\left[\frac{12.5779-10}{0.05}\right]\right)\\=&\$13,609.06.\end{aligned}$$

(b) The purchase price is $13,609.06(1.08)^{-10}=\$6,303.63$ $\blacksquare$

Example 32.4
Esther invests 100 at the end of each year for 12 years at an annual effective interest rate of i. The interest payments are reinvested at an annual effective rate of 5%. The accumulated value at the end of 12 years is 1748.40.
Calculate i.

Solution.
The equation of value at time $t = 12$ is

$$1748.40 = 1200 + 100i\left[\frac{s_{\overline{12}|0.05} - 12}{0.05}\right].$$

Solving this equation for i we find

$$i = \frac{[1748.40 - 1200](0.05)}{100[s_{\overline{12}|0.05} - 12]} = 7\% \blacksquare$$

If the payments of 1 are made at the beginning of each period (instead of at the end), the accumulated value at the end of the n periods is

$$AV = n + i(Is)_{\overline{n}|j} = n + i\left[\frac{s_{\overline{n+1}|j} - (n+1)}{j}\right].$$

Example 32.5
David pays 1,000 at the beginning of each year into a fund which earns 6%. Any interest earned is reinvested at 8%. Calculate the total that David will have at the end of 7 years.

Solution.
Note that payment are made at the beginning of the year (annuity-due). The answer is

$$AV = 7,000 + 1000(0.06)\left[\frac{\frac{1.08^8-1}{0.08} - 8}{0.08}\right] = 8,977.47 \blacksquare$$

An important consideration to a lender (investor) is the rate of repayment by the borrower. A faster rate of repayment by the borrower, results in a higher yield rate for the investor. This is illustrated in the following example.

Example 32.6
Three loan repayment plans are described for a $3,000 loan over a 6-year period with an effective rate of interest of 7.5%. If the repayments to the lender can be reinvested at an effective rate of

6%, find the yield rates (for the lender)
(a) if the entire loan plus accumulated interest is paid in one lump sum at the end of 6 years;
(b) if interest is paid each year as accrued and the principal is repaid at the end of 6 years;
(c) if the loan is repaid by level payments over the 6-year period.

Solution.
(a) The lump sum is $3000(1.075)^6 = \$4629.90$. Since there is no repayment to reinvest during the 6-year period, the yield rate is obviously 7.5%. This answer can be also found by solving the equation $3000(1+i)^6 = 4629.90$.
(b) At the end of each year during the 5-year period, the payment is

$$3000(0.075) = \$225.$$

The accumulated value of all payments at the end of the 6-year period is $3000 + 225s_{\overline{6}|0.06} = 3000+225(6.9753) = \$4569.44$. To find the yield rate, we solve the equation $3000(1+i)^6 = 4569.44 \rightarrow i = 0.07265 = 7.265\%$
(c) Each year during the 6-year period, the payment is R where $3000 = Ra_{\overline{6}|0.075} \rightarrow R = 639.13$. The accumulated value of all payments at the end of the 6-year period is $639.13s_{\overline{6}|0.06} = 639.13(6.9753) = \$4458.12$. To find the yield rate, we solve the equation $3000(1+i)^6 = 4458.12 \rightarrow i = 0.06825 = 6.825\%$ ■

Practice Problems

Problem 32.1
Vanessa invests \$5,000 at the end of each of the next 15 years and her investment earns interest at an annual effective rate of 10%. The interest that she receives at the end of each year is reinvested and earns interest at an annual effective rate of 5%. Calculate the accumulated value at time 15 years.

Problem 32.2
Lauren has 100,000 invested in a fund earning 5%. Each year the interest is paid to Lauren who is able to invest the interest at 8%. Calculate the amount that Lauren will have at the end of 10 years.

Problem 32.3
Julie has a sum of money, S, invested in a fund which earns 10%. Each year the fund pays the interest earned to Julie. Julie can only reinvest the interest at an annual effective interest rate of 6%. After 20 years, Julie has 100,000 total including the amount in the fund plus the reinvested interest. Calculate S.

Problem 32.4
Megan invests 500 at the end of each year. The investment earns 8% per year which is paid to Megan who reinvests the interest at 6%. Calculate how much Megan will have after 5 years.

Problem 32.5
Thomas invests X into Fund 1 at the beginning of each year for 10 years. Fund 1 pays interest annually into Fund 2. Fund 1 earns 7% annually while Fund 2 earns 6% annually. After 10 years, Thomas has a total of 50,000. Calculate X.

Problem 32.6
Chris is investing 1,000 at the beginning of each year into Fund A. Fund A earns interest at a nominal interest rate of 12% compounded monthly. Fund A pays Chris interest monthly. Chris reinvests that interest in Fund B earning an annual effective rate of 8%. Calculate the total amount in Fund A and Fund B after 10 years.

Problem 32.7
John invests 100 at the end of year one, 200 at the end of year two, etc until he invests 1,000 at the end of year ten. The investment goes into a bank account earning 4%. At the end of each year, the interest is paid into a second bank account earning 3%. Calculate the total amount John will have after 10 years.

Problem 32.8
Kathy pays 1,000 at the end of each year into Fund A which earns interest at an annual effective interest rate of i. At the end of each year, the interest earned is transferred to Fund B earning 10% interest. After 10 years. Kathy has 15,947.52. Calculate i.

Problem 32.9
It is desired to accumulate a fund of 1,000 at the end of 10 years by equal deposits at the beginning of each year. If the deposits earn interest at 8% effective, but the interest can only be reinvested at only 4% effective, show that the deposit necessary is

$$\frac{1000}{2s_{\overline{11}|0.04} - 12}.$$

Problem 32.10
A loan of 10,000 is being repaid with payments of 1,000 at the end of each year for 20 years. If each payment is immediately reinvested at 5% effective, find the yield rate of this investment.

Problem 32.11
An investor purchases a 5-year financial instrument having the following features:
(i) The investor receives payments of 1,000 at the end of each year for 5 years.
(ii) These payments earn interest at an effective rate of 4% per annum. At the end of the year, this interest is reinvested at the effective rate of 3% per annum.
Find the purchase price to the investor to produce a yield rate of 4%.

Problem 32.12
An investor deposits 1,000 at the beginning of each year for five years in a fund earning 5% effective. The interest from this fund can be reinvested at only 4% effective. Show that the total accumulated value at the end of ten years is

$$1250(s_{\overline{11}|0.04} - s_{\overline{6}|0.04} - 1)$$

Problem 32.13
A invests 2,000 at an effective interest rate of 17% for 10 years. Interest is payable annually and is reinvested at an effective rate of 11%. At the end of 10 years the accumulated interest is 5,685.48. B invests 150 at the end of each year for 20 years at an effective interest rate of 14%. Interest is payable annually and is reinvested at an effective rate of 11%. Find B's accumulated interest at the end of 20 years.

Problem 32.14 ‡
Victor invests 300 into a bank account at the beginning of each year for 20 years. The account pays out interest at the end of every year at an annual effective interest rate of i%. The interest is reinvested at an annual effective rate of $\left(\frac{i}{2}\right)$%.
The yield rate on the entire investment over the 20 year period is 8% annual effective. Determine i.

Problem 32.15 ‡
Sally lends 10,000 to Tim. Tim agrees to pay back the loan over 5 years with monthly payments payable at the end of each month.
Sally can reinvest the monthly payments from Tim in a savings account paying interest at 6%, compounded monthly. The yield rate earned on Sally's investment over the five-year period turned out to be 7.45%, compounded semi-annually.
What nominal rate of interest, compounded monthly, did Sally charge Tim on the loan?

Problem 32.16
Paul invests 1,000 at the beginning of each year for 10 years into a Fund A earning Y% each year. At the end of the year, any interest earned is moved to Fund B which earns 5%. Combining both Funds, Paul has a total of 11,924.08 at the end of 10 years. Determine Y.

Problem 32.17
Lisa invests 1200 at the beginning of each year for 8 years into an account earning 8%. The interest earned each year is transferred to an account earning 6%. At the end of 8 years, the total amount is paid out.
Calculate the amount an investor would pay now for the final payout if the investor wanted a return of 10%.

Problem 32.18
Amy invests 1000 at an effective annual rate of 14% for 10 years. Interest is payable annually and is reinvested at an annual effective rate of i. At the end of 10 years the accumulated interest is 2341.08.
Bob invests 150 at the end of each year for 20 years at an annual effective rate of 15%. Interest is payable annually and is reinvested at an annual effective rate of i.
Find Bob's accumulated interest at the end of 20 years.

Problem 32.19 ‡
An investor pays \$100,000 today for a 4-year investment that returns cash flows of \$60,000 at the end of each of years 3 and 4. The cash flows can be reinvested at 4.0% per annum effective. If the rate of interest at which the investment is to be valued is 5.0%, what is the net present value of this investment today?

Problem 32.20 ‡
An investor wishes to accumulate 10,000 at the end of 10 years by making level deposits at the beginning of each year. The deposits earn a 12% annual effective rate of interest paid at the end of each year. The interest is immediately reinvested at an annual effective interest rate of 8%. Calculate the level deposit.

Problem 32.21 ‡
Payments of X are made at the beginning of each year for 20 years. These payments earn interest at the end of each year at an annual effective rate of 8%. The interest is immediately reinvested at an annual effective rate of 6%. At the end of 20 years, the accumulated value of the 20 payments and the reinvested interest is 5600. Calculate X.

Problem 32.22
A deposit of 1 is made at the end of each year for 30 years into a bank account that pays interest at the end of each year at j per annum. Each interest payment is reinvested to earn an annual effective interest rate of $\frac{j}{2}$. The accumulated value of these interest payments at the end of 30 years is 72.88. Determine j.

Problem 32.23 ‡
Susan invests Z at the end of each year for seven years at an annual effective interest rate of 5%. The interest credited at the end of each year is reinvested at an annual effective rate of 6%. The accumulated value at the end of seven years is X.
Lori invests Z at the end of each year for 14 years at an annual effective interest rate of 2.5%. The interest credited at the end of each year is reinvested at an annual effective rate of 3%. The accumulated value at the end of 14 years is Y.
Calculate Y/X.

Problem 32.24 ‡
Jason deposits 3960 into a bank account at $t = 0$. The bank credits interest at the end of each year at a force of interest

$$\delta_t = \frac{1}{8+t}.$$

Interest can be reinvested at an annual effective rate of 7%. The total accumulated amount at time $t = 3$ is equal to X. Calculate X.

Problem 32.25 ‡
Eric deposits 12 into a fund at time 0 and an additional 12 into the same fund at time 10. The fund credits interest at an annual effective rate of i. Interest is payable annually and reinvested at an annual effective rate of $0.75i$. At time 20 the accumulated amount of the reinvested interest payments is equal to 64. Calculate $i, i > 0$.

Problem 32.26 ‡
At time t = 0, Sebastian invests 2000 in a fund earning 8% convertible quarterly, but payable annually. He reinvests each interest payment in individual separate funds each earning 9% convertible quarterly, but payable annually. The interest payments from the separate funds are accumulated in

a side fund that guarantees an annual effective rate of 7%. Determine the total value of all funds at $t = 10$.

Problem 32.27
A loan of 1000 is being repaid in 10 years by semiannual installments of 50, plus interest on the unpaid balance at 4% per annum compounded semiannually. The installments and interest payments are reinvested at 5% per annum compounded semiannually. Calculate the annual effective yield rate of the loan.

Problem 32.28
John invests a total of 10,000. He purchases an annuity with payments of 1000 at the beginning of each year for 10 years at an effective annual interest rate of 8%. As annuity payments are received, they are reinvested at an effective annual interest rate of 7%.
The balance of the 10,000 is invested in a 10-year certificate of deposit with a nominal annual interest rate of 9%, compounded quarterly.
Calculate the annual effective yield rate on the entire 10,000 investment over the 10 year period.

Problem 32.29
Brown deposits $1000 at the beginning of each year for 5 years, but makes no additional deposits in years 6 through 10. The fund pays interest annually at an annual effective interest rate of 5%. Interest is reinvested at only a 4% annual effective interest rate. What is Brown's total accumulated value at the end of 10 years?

Problem 32.30
You have $20,000 to invest. Alternative A allows you to invest your money at 8% compounded annually for 12 years. Alternative B pays you simple interest at the end of each year of 6%. At what rate must you reinvest the interest in alternative B to have the same accumulation after 12 years?

Problem 32.31
Henry invests 2000 at the beginning of the year in a fund which credits interest at an annual effective rate of 9%. Henry reinvests each interest payment in a separate fund, accumulating at an annual effective rate of 8%. The interest payments from this fund accumulate in a bank account that guarantees an annual effective rate of 7%. Determine the sum of the principal and interest at the end of 10 years.

Problem 32.32
Jim borrowed 10,000 from Bank X at an annual effective rate of 8%. He agreed to repay the bank with five level annual installments at the end of each year. At the same time, he also borrowed

15,000 from Bank Y at an annual effective rate of 7.5%. He agreed to repay this loan with five level annual installments at the end of each year. He lent the 25,000 to Wayne immediately in exchange for four annual level repayments at the end of each year, at an annual effective rate of 8.5%. Jim can only reinvest the proceeds at an annual effective rate of 6%. Immediately after repaying the loans to the banks in full, determine how much Jim has left.

33 Interest Measurement of a Fund: Dollar-Weighted Interest Rate

An **investment fund** usually is a firm that invests the pooled funds of investors for a fee. In this section we examine a method of determining the interest rate earned by an investment fund.
In practice, it is common for a fund to be incremented with new principal deposits, decremented with principal withdrawals, and incremented with interest earnings many times throughout a period. Since these occurrences are often at irregular intervals, we devise notation for the purpose of obtaining the effective rate of interest i over one measurement period:

A = the amount in the fund at the beginning of the period,i.e. $t = 0$.
B = the amount in the fund at the end of the period, i.e. $t = 1$.
I = the amount of interest earned during the period.
c_t = the net amount of principal contributed at time t (inflow less outflow at time t) where $0 \leq t \leq 1$.
C = $\sum_t c_t$ = total net amount of principal contributed during the period (if negative, this is an indication of net withdrawal).

The amount at the end of a period is equal to the amount at the beginning of the period plus the net principal contributions plus the interest earned:

$$B = A + C + I. \tag{33.1}$$

To be consistent with the definition of the effective rate we will assume that all the interest earned I is received at the end of the period. Assuming compound interest i throughout the period, the exact equation of value for the interest earned over the period $0 \leq t \leq 1$ is

$$I = iA + \sum_{0 \leq t \leq 1} c_t[(1+i)^{1-t} - 1]. \tag{33.2}$$

That is, the amount of interest earned is the sum of amounts of interest earned by each individual contribution, plus the amount of interest earned on the balance at the beginning. Note that $(1+i)^{1-t} - 1$ is the effective rate for period from t to 1,i.e.,

$$\frac{A(1) - A(t)}{A(t)} = \frac{A(1)}{A(t)} - 1 = (1+i)^{1-t} - 1.$$

Substituting equation(33.2) into equation (33.1) we find the equation of value

$$B = A(1+i) + \sum_{0 \leq t \leq 1} c_t(1+i)^{1-t}. \tag{33.3}$$

The interest i satisfying equation (33.3) is called the **dollar-weighted rate of interest**. Note that the required return of the investment, cost of capital and dollar weighted rate of return stand for the same quantity.

Example 33.1
At the beginning of a year, an investment fund was established with an initial deposit of \$3,000. At the end of six months, a new deposit of \$1,500 was made. Withdrawals of \$500 and \$800 were made at the end of four months and eight months respectively. The amount in the fund at the end of the year is \$3,876. Set up the equation of value to calculate the dollar-weighted rate of interest.

Solution.
The equation of value is

$$3,000(1+i)+1,500(1+i)^{0.5}-500(1+i)^{\left(1-\frac{4}{12}\right)}-800(1+i)^{\left(1-\frac{8}{12}\right)}=3,876 \blacksquare$$

Finding i from the equation (33.2) requires approximation methods and this is not an easy problem. However, in practice one uses a simple interest approximation

$$(1+i)^{1-t}-1=1+(1-t)i-1=(1-t)i$$

obtaining

$$I\approx iA+\sum_{0\leq t\leq 1}c_t(1-t)i$$

which leads to the approximation

$$i\approx \frac{I}{A+\sum_{0\leq t\leq 1}c_t(1-t)}. \tag{33.4}$$

It is shown that this approximation is very good as long as the c_t's are small in relation to A which is often the case in practice. However, if the c_t's are not small in relation to A, then the error can become significant. The denominator in the above approximation is commonly called the **exposure associated with** i.
Alternatively, by using equation (33.3) one finds i by solving the equation

$$B=A(1+i)+\sum_{0\leq t\leq 1}c_t[1+(1-t)i].$$

Example 33.2
Using Example 33.1, find the approximate effective rate of interest earned by the fund during the year using the dollar-weighted rate of return formula.

Solution.
The interest earned I is computed to be

$$I = 3876 + 500 + 800 - (3000 + 1500) = 676.$$

The exposure associated with i is

$$3000 + (-500)\left(1 - \frac{1}{3}\right) + (1500)\left(1 - \frac{1}{2}\right) + (-800)\left(1 - \frac{2}{3}\right) = 3150.$$

Thus, the approximate effective rate is

$$\frac{676}{3150} = 0.2146 = 21.46\%.$$

Alternatively, i can be found by solving the equation

$$3,000(1+i) + 1,500(1+0.5i) - 500(1+\frac{2}{3}i) - 800(1+\frac{i}{3}) = 3,876$$

giving $i = 21.46\%$ ■

Formula (33.4) is in a form which can be directly calculated. However, it is tedious to use it because of the summation term in the denominator. Therefore, estimating the summation is favored. One way for doing that is to assume that the net principal contributions C occur at time k that can be approximated by the arithmetic weighted average

$$k = \frac{1}{C}\sum_{0\le t\le 1} t \cdot c_t$$

obtaining a simpler approximation

$$\begin{aligned} i \approx & \frac{I}{A + (1-k)C} = \frac{I}{A + (1-k)(B - A - I)} \\ = & \frac{I}{kA + (1-k)B - (1-k)I}. \end{aligned}$$

Example 33.3
A fund has 100,000 on January 1 and 125,000 on December 31. Interest earned by the fund during the year totaled 10,000. Calculate the net yield earned by the fund assuming that the net contributions occurred on April 1.

Solution.
Using the formula, $i \approx \frac{I}{[kA+(1-k)B-(1-k)I]}$, we have

$$i \approx \frac{10,000}{[(1/4)100,000 + (3/4)125,000 - (3/4)10,000]} = \frac{10,000}{111,250} = 8.98876404\% \blacksquare$$

One useful special case is when the principal deposits or withdrawals occur uniformly throughout the period. Thus, on average, we might assume that net principal contributions occur at time $k = \frac{1}{2}$, in which case we have

$$i \approx \frac{I}{0.5A + 0.5B - 0.5I} = \frac{2I}{A+B-I}. \tag{33.5}$$

This formula has been used for many years by insurance company regulators in both Canada and the US to allow them to get a feel for the investment returns a particular company is obtaining.

Example 33.4
You have a mutual fund. Its balance was \$10,000 on 31/12/1998. You made monthly contributions of \$100 at the start of each month and the final balance was \$12,000 at 31/12/1999. What was you approximate return during the year?

Solution.
We are given that $A = \$10,000$, $B = \$12,000$, and $C = \$1,200$. Thus, $I = B - (A+C) = \$800$ and

$$i = \frac{2(800)}{10,000 + 12,000 - 800} = 7.54717\% \blacksquare$$

Example 33.5
Find the effective rate of interest earned during a calendar year by an insurance company with the following data:

Assets, beginning of year	10,000,000
Premium income	1,000,000
Gross investment income	530,000
Policy benefits	420,000
Investment expenses	20,000
Other expenses	180,000

Solution.
We are given

$$\begin{aligned}
A =&10,000,000\\
B =&10,000,000+1,000,000+530,000-(420,000+20,000+180,000)\\
=&10,910,000\\
I =&530,000-20,000=510,000
\end{aligned}$$

Thus,

$$i=\frac{2I}{A+B-I}=\frac{2(510,000)}{10,000,000+10,910,000-510,000}=5\% \blacksquare$$

Practice Problems

Problem 33.1
A fund has 10,000 at the start of the year. During the year \$5,000 is added to the fund and \$2,000 is removed. The interest earned during the year is \$1,000. Which of the following are true?
(I) The amount in the fund at the end of the year is \$14,000.
(II) If we assume that any deposits and withdrawals occur uniformly throughout the year, i is approximately 8.33%.
(III) If the deposit was made on April 1 and the withdrawal was made on August 1, then i is approximately 7.74%.

Problem 33.2
The funds of an insurance company at the beginning of the year were \$500,000 and at the end of the year \$680,000. Gross interest earned was \$60,000, against which there were investment expenses of \$5,000. Find the net effective rate of interest yielded by the fund.

Problem 33.3
A fund earning 4% effective has a balance of \$1,000 at the beginning of the year. If \$200 is added to the fund at the end of three months and if \$300 is withdrawn from the fund at the end of nine months, find the ending balance under the assumption that simple interest approximation is used.

Problem 33.4
At the beginning of the year an investment fund was established with an initial deposit of \$1,000. A new deposit of \$500 was made at the end of four months. Withdrawals of \$200 and \$100 were made at the end of six months and eight months, respectively. The amount in the fund at the end of the year is \$1,272. Find the approximate effective rate of interest earned by the fund during the year using the dollar-weighted rate of return formula.

Problem 33.5 ‡
An association had a fund balance of 75 on January 1 and 60 on December 31. At the end of every month during the year, the association deposited 10 from membership fees. There were withdrawals of 5 on February 28, 25 on June 30, 80 on October 15, and 35 on October 31.
Calculate the dollar-weighted rate of return for the year.

Problem 33.6 ‡
An insurance company earned a simple rate of interest of 8% over the last calendar year based on the following information:

Assets, beginning of year	25,000,000
Sales revenue	X
Net investment income	2,000,000
Salaries paid	2,200,000
Other expenses paid	750,000

All cash flows occur at the middle of the year. Calculate the effective yield rate.

Problem 33.7 ‡
At the beginning of the year, an investment fund was established with an initial deposit of 1,000. A new deposit of 1,000 was made at the end of 4 months. Withdrawals of 200 and 500 were made at the end of 6 months and 8 months, respectively. The amount in the fund at the end of the year is 1,560.
Calculate the dollar-weighted yield rate earned by the fund during the year.

Problem 33.8
On January 1, an investment account is worth \$500,000. On April 1, the account value has increased to \$530,000, and \$120,000 of new principal is deposited. On August 1, the account value has decreased to \$575,000, and \$250,000 is withdrawn. On January 1 of the following year, the account value is \$400,000. Compute the yield rate using the dollar-weighted method.

Problem 33.9
A fund has 10,000 at the beginning of the year and 12,000 at the end of the year. Net contributions of 1,000 were made into the fund during year. Calculate the net yield earned by the fund assuming that the net contributions were contributed uniformly throughout the year.

Problem 33.10
A fund has 100,000 on January 1 and 125,000 on December 31. Interest earned by the fund during the year totaled 10,000. The net yield earned by the fund during the year was 9.6385%. Two contributions were made to the fund during the year and there were no withdrawals. The contributions were for equal amounts made two months apart.
Determine the date of the first contribution.

Problem 33.11
An investor fund has a balance on January 1 of \$273,000 and a balance on December 31 of \$372,000. The amount of interest earned during the year was \$18,000 and the computed yield rate on the fund was 6%. What was the average date for contributions to and withdrawals from the fund?

Problem 33.12
You deposited 5,000 into a fund on January 1, 2002. On March 1,2002, you withdrew 1,000 from the fund. On August 1, 2002, you deposited 700 into the fund. On October 1, 2002, you deposited another 750 into the fund. If your annual dollar-weighted rate of return on your investment was 12.00%, what was your fund worth on December 31, 2002?

Problem 33.13
At the beginning of the year a fund has deposits of 10,000. A deposit of 1000 is made after 3 months and a withdrawal of 2000 is made after 9 months. The amount in the fund at the end of the year is 9500. Find the dollar weighted rate of interest
(a) using formula (33.3)
(b) using formula (33.4)
(c) using formula (33.5).

Problem 33.14
Deposits of 10000 are made into an investment fund at time zero and 1. The fund balance at time 2 is 21000. Compute the effective yield using the dollar weighted method.

Problem 33.15
An investor puts \$100 into a mutual fund in the first year and \$50 in the second year. At the end of the first year the mutual pays a dividend of \$10. The investor sells the holdings in the mutual fund at the end of the second year for \$180. Find the dollar weighted return.

Problem 33.16
A fund has 100,000 at the start of the year. Six months later, the fund has a value of 75,000 at which time, Stuart adds an additional 75,000 to the fund. At the end of the year, the fund has a value of 200,000.
Calculate the exact dollar weighted rate of return(i.e. using (33.3)).

Problem 33.17
A fund has \$2,000,000 at the beginning of the year. On June 1 and September 1, two withdrawals of \$100,000 each were made. On November 1, a contribution of \$20,000 was made. The value in the fund at the end of the year is \$1,900,000.
Find
(a) the exact dollar weighted rate of return;
(b) the dollar weigthed rate of return using formula (33.4).

Problem 33.18
The following table shows the history of account balances, deposits, and withdrawals for an account

during 2008:

Date	Value before dep/withd	Dep/Withd
January 1,2008	10,000	0
March 1,2008	9,500	3,000
June 1,2008	14,000	$-2,000$
September 1,2008	12,000	$-2,000$
January 1,2009	12,000	

What is this account's dollar-weighted rate of return?(Assume 30-day months.)

Problem 33.19
On January 1, 2000, the balance in account is \$25200. On April 1, 2000, \$500 are deposited in this account and on July 1, 2001, a withdraw of \$1000 is made. The balance in the account on October 1, 2001 is \$25900. What is the annual rate of interest in this account according with the dollarweighted method?

Problem 33.20
You invest \$10,000 in a fund on January 1, 2008. On May 1, 2008, you withdraw \$3,000 from the fund. On November 1, 2008, you deposit \$4,000 into the fund. On December 31, 2008, your fund is worth \$13,500. What was the annual dollar-weighted rate of return on your investment?

Problem 33.21
You invest \$5,000 in a fund on 1/1/07. On 3/1/07, you withdraw X from the fund. On 7/1/07, you deposit \$2,500 into the fund. On 9/1/07, you withdraw \$1,000 from the fund. On 12/31/07, your fund is worth \$5,700. The annual dollar−weighted rate of return on your investment was 12.0%. Find X.

34 Interest Measurement of a Fund: Time-Weighted Rate of Interest

The dollar-weighted rate of interest depends on the precise timing and amount of the cash flows. In practice, professional fund managers who direct investment funds have no control over the timing or amounts of the external cash flows. Therefore, if we are comparing the performance of different fund managers, the dollar weighted rate of interest doesn't always provide a fair comparison. In this section, we consider an alternative measure of performance that does not depend on the size or the timing of the cash flows, namely the **time-weighted rate of interest** also known as the **time-weighted rate of return.**

Consider the following one-year investment. An amount X is invested in a fund at the beginning of the year. In six months, the fund is worth $\frac{X}{2}$ at which time the investor can decide to add to the fund, to withdraw from the fund, or to do nothing. At the end of the year, the fund balance is double the balance at six months.

We next examine three situations related to the above investment: Consider three investors A, B, and C. Investor A initially invests $X = \$1,000$. At the end of six months his investment is worth \$500. He decideds not to withdraw from the fund or deposits into the fund. His fund at the end of the year worths \$1,000. Since the interest earned is $I = 0$, the dollar-weighted rate of interest is $i = 0$.

Next, investor B invests initially $X = \$1,000$. Again, after six months the balance is \$500. At the end of six months the investor withdraws half the fund balance (i.e. \$250). His balance in the fund at the end of the year is \$500. The equation of value at $t = 1$ for this transaction is $1000(1+i) - 250(1+i)^{\frac{1}{2}} = 500$. Solving this quadratic equation we find $(1+i)^{\frac{1}{2}} = 0.84307$ and this implies a yield rate of $i = -0.2892$ or -28.92%.

Finally, investor C invests initially $X = \$1,000$ and at six months deposits an amount equal to the fund balance(i.e. \$500). Hence, his balance at the end of the year is \$2,000. The equation of value at time $t = 1$ is $1000(1+i) + 500(1+i)^{\frac{1}{2}} = 2000$. Solving this quadratic equation we find $(1+i)^{\frac{1}{2}} = 1.18614$. Hence, the yield rate in this case is $i = 0.4069$ or 40.69%.

The yield rate for investor C is so much better than the yield rate for either investor A or investor B, because of the decisions made by each investor.

The dollar-weighted rate of interest measures both the behavior of the fund and the skills of the investor. If the investor did not perform any transactions, then the dollar-weighted rate of interest would be close to 0%.

Now how can we evaluate the decisions made by the fund manager? In the illustration above, we find the yield rate for the first six months to be $j_1 = -50\%$ and for the second six months to be $j_2 = 100\%$. If i is the yield rate for the entire year then

$$1 + i = (1 + j_1)(1 + j_2) = 1$$

which implies that $i = 0$ and this is regardless of when cash is deposited or withdrawn. This indicates that the manager did a poor job of maintaining the fund.
Yield rates computed by considering only changes in interest rate over time (which is what was previously done to evaluate the fund manager's performance) are called **time-weighted rates of interest.**
We can generalize the above approach as follows. Suppose $m-1$ deposits or withdrawals are made during a year at times $t_1, t_2, \cdots, t_{m-1}$ (so that no contributions at $t_0 = 0$ and $t_m = 1$). Thus, the year is divided into m subintervals. For $k = 1, 2, \cdots, m$ we let j_k be the yield rate over the kth subinterval. For $k = 1, \cdots, m-1$, let C_{t_k} be the net contribution at exact time t_k and B_{t_k} the value of the fund before the contribution at time t_k. Note that $C_0 = C_m = 0$ and B_0 is the initial investment and B_1 is the value of the fund at the end of the year.
The yield rate of the fund from time t_{k-1} to time t_k satisfies the equation of value

$$B_{t_k} = (1 + j_k)(B_{t_{k-1}} + C_{t_{k-1}})$$

or

$$1 + j_k = \frac{B_{t_k}}{B_{t_{k-1}} + C_{t_{k-1}}}, \quad k = 1, 2, \cdots, m.$$

The overall yield rate i for the entire year is given by

$$1 + i = (1 + j_1)(1 + j_2) \cdots (1 + j_m)$$

or

$$i = (1 + j_1)(1 + j_2) \cdots (1 + j_m) - 1.$$

We call i the **time-weighted rate of return.**

Example 34.1
Suppose that an investor makes a series of payments and withdrawals, as follows:

Date	Flow	Balance before	Balance after
01/01/2003	0	100,000	100,000
30/06/2003	+1,000,000	74,681	1,074,681
31/12/2003	0	1,474,081	1,474,081

(a) Compute the dollar-weighted rate of interest that the investor has realized.
(b) Compute the time-weigted rate of interest.

Solution.
(a) We assume that the contribution of \$1,000,000 occurred exactly in the middle of the year. We

have $A = \$100,000,\ B = \$1,474,081,\ C = \$1,000,000$. Hence, $I = 1,474,081 - 1,000,000 - 100,000 = 374,081$ and

$$i = \frac{2(374,081)}{100,000 + 1,474,081 - 374,081} = 62.35\%.$$

(b) We have

$$i = \left(\frac{74,681}{100,000}\right)\left(\frac{1,474,081}{1,074,681}\right) - 1 = 2.44\% \blacksquare$$

It follows from the previous example that the time-weighted method does not provide a valid measure of yield rate as does the dollar weighted method. However, the time-weighted method does provide a better indicator of underlying investment performance than the dollar-weighted method.

Example 34.2

Your balance at time 0 is \$2,000. At time $t = \frac{1}{3}$, the balance has grown to \$2,500 and a contribution of \$1,000 is made. At time $t = \frac{2}{3}$, the balance has dropped to \$3,000 and a withdrawal of \$1,500 is made. At the end of the year, the fund balance is \$2,000. What is the time weighted rate of return?

Solution.

We create the following chart

k	t_k	B_{t_k}	C_{t_k}	j_k
0	0.0000	\$2,000	\$0	0%
1	1/3	\$2,500	\$1,000	25.00%
2	2/3	\$3,000	-\$1,500	-14.29%
3	1	\$2,000	\$0	33.33%

The time weighted rate of return is

$$i = (1.25)(.8571)(1.3333) - 1 = .428464 \blacksquare$$

Example 34.3

You are given the following information about an investment account:

Date	June 30,01	Sep. 30,01	Dec. 31,01	March 31,02	June 30,02
B_k	12,000	10,000	14,000	13,000	X
Deposit		2,000		2,000	
Withdrawal	2,000		2,000		

If the effective annual dollar-weighted rate of return from June 30, 2001 to June 30, 2002 was exactly 10%, what was the effective annual time-weighted rate of return during that same period?

Solution.
First, we find X :

$$X = 10,000(1.1) + 2,000(1.1)^{0.75} - 2,000(1.1)^{0.5} + 2,000(1.1)^{0.25} = 13,098.81$$

The time-weighted rate of return is then

$$i = \frac{10,000}{12,000 - 2,000} \cdot \frac{14,000}{10,000 + 2,000} \cdot \frac{13,000}{14,000 - 2,000} \cdot \frac{13,098.81}{13,000 - 2,000} - 1 = 10.37\% \blacksquare$$

Practice Problems

Problem 34.1
Which of the following are true?
(I) Time-weighted method provides better indicator for the fund underlying performance than the dollar-weighted method.
(II) Time weighted rates of interest will always be higher than dollar weighted rates of interest.
(III) Dollar weighted rate of interest provide better indicators of underlying investment performance than do time weighted rates of interest.
(IV) Dollar weighted rates of interest provide a valid measure of the actual investment results.

Problem 34.2
A fund has 1,000 at beginning of the year. Half way through the year, the fund value has increased to 1,200 and an additional 1,000 is invested. At the end of the year, the fund has a value of 2,100.
(a) Calculate the exact dollar weighted rate of return using compound interest.
(b) Calculate the estimated dollar weighted rate of return using the simple interest assumptions.
(c) Calculate the time weighted rate of return.

Problem 34.3
A fund has 100,000 at the start of the year. Six months later, the fund has a value of 75,000 at which time, Stuart adds an additional 75,000 to the fund. At the end of the year, the fund has a value of 200,000.
Calculate the time weighted rate of return.

Problem 34.4
A fund has 100,000 at the start of the year. Six months later, the fund has a value of 75,000 at which time, Stuart adds an additional 75,000 to the fund. At the end of the year, the fund has a value of 200,000.
Calculate the exact dollar weighted rate of return.

Problem 34.5
A fund has 10,000 at the start of the year. Six months later, the fund has a value of 15,000 at which time, Stuart removes 5,000 from the fund. At the end of the year, the fund has a value of 10,000.
Calculate the exact dollar weighted rate of return less the time weighted rate of return.

Problem 34.6 ‡
You are given the following information about an investment account:

Date	Value Immediately Before Deposit	Deposit
January 1	10	
July 1	12	X
December 31	X	

Over the year, the time-weighted return is 0%, and the dollar-weighted (money-weighted) return is Y. Determine Y.

Problem 34.7 ‡
An investor deposits 50 in an investment account on January 1 . The following summarizes the activity in the account during the year:

Date	Value Immediately Before Deposit	Deposit
March 15	40	20
June 1	80	80
October 1	175	75

On June 30, the value of the account is 157.50 . On December 31, the value of the account is X. Using the time-weighted method, the equivalent annual effective yield during the first 6 months is equal to the (time-weighted) annual effective yield during the entire 1-year period.
Calculate X.

Problem 34.8
On January 1, an investment account is worth \$100,000. On May 1, the account value has increased to \$112,000, and \$30,000 of new principal is deposited. On November 1, the account value has decreased to \$125,000, and \$42,000 is withdrawn. On January 1 of the following year, the account value is \$100,000. Compute the yield rate using
(a) the dollar-weighted method and
(b) the time-weighted method.

Problem 34.9
In Problem 34.8, change May 1 to June 1 and November 1 to October 1.
(a) Would the yield rate change when computed by the dollar-weighted method?
(b) Would the yield rate change when computed by the time-weighted method?

Problem 34.10
In Problem 34.8, assume that everything is unchanged except that an additional \$5,000 is withdrawn on July 1.
(a) Would the yield rate change when computed by the dollar-weighted method?
(b) Explain why the yield rate cannot be computed by the time-weighted method.

Problem 34.11
A mutual fund account has the balance \$100 on January 1. On July 1 (exactly in the middle of the year) the investor deposits \$250 into the account. The balance of the mutual fund at the end of the year is \$400.
(a) Calculate the exact dollar-weighted annual rate of interest for the year.
(b) Suppose that the balance of the fund immediately before the deposit on July 1 is \$120. Calculate the time-weighted annual rate of interest for the year.

Problem 34.12
You are given the following information about two funds:
Fund X: January 1, 2005: Balance 50,000
May 1, 2005: Deposit 24,000; Pre-deposit Balance 50,000
November 1, 2005: Withdrawal 36,000; Pre-withdrawal Balance 77,310
December 31, 2005: Balance 43,100

Fund Y: January 1, 2005: Balance 100,000
July 1, 2005: Withdrawal 15,000; Pre-withdrawal Balance 105,000
December 31, 2005: Balance F.

Fund Y's time-weighted rate of return in 2005 is equal to Fund X's dollar-weighted rate of return in 2005. Calculate F.

Problem 34.13
You invested 10,000 in a fund on January 1, 2000. On April 1, 2000, your fund was worth 11,000, and you added 2,000 to it. On June 1, 2000, your fund was worth 12,000 and you withdrew 2,000 from the fund. On December 31, 2000, your fund was worth 9,500. What was the time-weighted rate of return on your investment from January 1, 2000 to December 31, 2000?

Problem 34.14
Deposits of \$1000 are made into an investment fund at time 0 and 1. The fund balance is \$1200 at time 1 and \$2200 at time 2.
(a) Compute the annual effective yield rate computed by a dollar-weighted method.
(b) Compute the annual effective yield rate which is equivalent to that produced by a time-weighted method.

Problem 34.15
Let A be the fund balance on January 1, B the fund balance on June 30, and C the balance on December 31.

(a) If there are no deposits or withdrawals, show that yield rates computed by the dollar-weighted method and the time-weighted method are both equal.
(b) If there was a single deposit of D immediately after the June 30 balance was calculated, find expressions for the dollar-weighted and time-weighted yield rates.
(c) If there was a single deposit of D immediately before the June 30 balance was calculated, find expressions for the dollar-weighted and time-weighted yield rates.
(d) Compare the dollar-weighted yield rates in (b) and (c).
(e) Compare the time-weighted yield rates in (b) and (c).

Problem 34.16
100 is deposited into an investment account on January 1, 1998. You are given the following information on investment activity that takes place during the year:

	April 19,1998	October 30,1998
Value immediately prior to deposit	95	105
deposit	2X	X

The amount in the account on January 1, 1999 is 115. During 1998, the dollar-weighted return is 0% and the time-weighted return is Y. Calculate Y.

Problem 34.17
An investment manager's portfolio begins the year with a value of 100,000. Eleven months through the year a withdrawal of 50,000 is made and the value of the portfolio after the withdrawal is 57,000. At the end of the year the value of the portfolio is 60,000. Find the time-weighted yield rate less the dollar-weighted yield rate.

Problem 34.18
On January 1, 1997 an investment account is worth 100,000. On April 1, 1997 the value has increased to 103,000 and 8,000 is withdrawn. On January 1, 1999 the account is worth 103,992. Assuming a dollar weighted method for 1997 and a time weighted method for 1998 the annual effective interest rate was equal to x for both 1997 and 1998. Calculate x.

Problem 34.19
On January 1, 1999, Luciano deposits 90 into an investment account. On April 1, 1999, when the amount in Luciano's account is equal to X, a withdrawal of W is made. No further deposits or withdrawals are made to Luciano's account for the remainder of the year. On December 31, 1999, the amount in Luciano's account is 85. The dollar weighted return over the 1 year period is 20%. The time weighted return over the 1 year period is 16%. Calculate X.

Problem 34.20
You are given the following information about the activity in two different investment accounts.

Account K Activity

Date	Fund value before activity	Deposit	Withdrawal
January 1,1999	100		
July 1,1999	125		X
October 1,1999	110	2X	
December 31,1999	125		

Account L Actyivity

Date	Fund value before activity	Deposit	Withdrawal
January 1,1999	100		
July 1,1999	125		X
December 31,1999	105.8		

During 1999 the dollar weighted return for investment account K equals the time weighted return for investment account L, which equals i. Calculate i.

Problem 34.21
The following table shows the history of account balances, deposits, and withdrawals for an account during a particular year:

Date	Value before dep/withd	Dep/Withd
January 1,X	1,000	0
May 1,X	1,050	1,100
September 1,X	2,250	-900
January 1,X+1	16,00	

What is the relationship among this account's dollar-weighted rate of return (DW), its time-weighted rate of return (TW), and its internal rate of return (IRR)?Assume 30-day months.

35 Allocating Investment Income: Portfolio and Investment Year Methods

Consider an investment fund that involves many investors with seperate accounts. An example would be a pension fund for retirement in which each plan participant has an account but with commingled pool of assets. Each investor owns shares in the fund.
In this section we consider the issue of allocating investment income to the various accounts. Two common methods are in use: the **portfolio method** and the **investment year method** also known as the **new money method.**

The Portfolio Method
Under the **portfolio method** an average rate based on the earnings of the entire fund is computed and credited to each account. All new deposits will earn this same portfolio rate.
There are disadvantages of this method during time of fluctuating interest rates. For example, suppose that the market rates have been rising significantly during the recent years while the average portfolio rate is being lowered. In this case, there is a higher tendency for less new deposits to the fund and more withdrawals from the fund.
The investment year method is an attempt to address this problem and to allocate investment income in a more equitable manner.

Investment Year Method (IYM)
We will describe this method on an annual basis, although typically smaller time intervals would be used. Under the **investment year method,** an investors's contribution will be credited during the year with the interest rate that was in effect at the time of the contribution. This interest rate is often referred to as the **new money rate.**
For example, a portion of the IYM chart for a fund might look something like the one shown in Table 35.1.
Suppose we invested \$1,000 on January 1, 1994 and \$500 on January 1, 1995. Then our total accumulation on Jan. 1, 2003 would be

$$(1.064)(1.068)(1.071)(1.069)(1.073)(1.051)(1.049)(1.048)(1.047)1000 = 1688.75$$

from our 1994 contribution, plus

$$(1.069)(1.070)(1.070)(1.074)(1.050)(1.049)(1.048)(1.047)500 = 794.31$$

from our 1995 contribution, for a total of

$$1688.75 + 794.31 = 2483.06.$$

Purchase Year	1994	1995	1996	1997	1998	1999	2000	2001	2002
1994	6.4%	6.8%	7.1%	6.9%	7.3%	5.1%	4.9%	4.8%	4.7%
1995		6.9%	7.0%	7.0%	7.4%	5.0%	4.9%	4.8%	4.7%
1996			7.1%	7.3%	7.3%	5.5%	5.4%	4.8%	4.7%
1997				7.0%	7.4%	5.4%	5.2%	4.6%	4.7%
1998					7.2%	5.7%	5.5%	4.5%	4.4%
1999						5.8%	5.1%	4.3%	4.7%
2000							5.0%	4.1%	4.6%
2001								4.0%	4.5%
2002									4.1%

Table 35.1

Notice that from the chart above, investments made between 1994-1996 all earned the same rate of return both in 2001 (4.8%) and 2002 (4.7%). This illustrates the principle that under the investment year method, funds on investment longer than a certain fixed period (5 years in our example) are assumed to grow at the overall yield rate for the fund (the portfolio rate), regardless of when the funds were invested.
To better indicate the switch from the investment year method to the portfolio method, the data from the chart above would typically be displayed by a two-dimensional table such as the one in Table 35.2 If y is the calendar year of deposit, and m is the number of years for which the investment year method is applicable, then the rate of interest credited for the t^{th} year of investment is denoted as i_t^y for $t = 1, 2, \cdots, m$.
For $t > m$, the portfolio method is applicable, and the portfolio rate of interest credited for calendar year y is denoted as i^y.
The investment year method is more complicated than the portfolio method, but it was deemed necessary to attract new deposits and to discourage withdrawals during periods of rising interest rates.
We illustrate the use of the investment year method in the following example.

Example 35.1
You are given the following table of interest rates (in percentages). A person deposits 1,000 on January 1, 1997. Find the accumulated amount on January 1, 2000
(a) under the investment year method;
(b) under the portfolio method;
(c) when the balance is withdrawn at the end of every year and is reinvested at the new money rate.

y	i_1^y	i_2^y	i_3^y	i_4^y	i_5^y	i^{y+5}	Portfolio Year
1992	8.25	8.25	8.4	8.5	8.5	8.35	1997
1993	8.5	8.7	8.75	8.9	9.0	8.6	1998
1994	9.0	9.0	9.1	9.1	9.2	8.85	1999
1995	9.0	9.1	9.2	9.3	9.4	9.1	2000
1996	9.25	9.35	9.5	9.55	9.6	9.35	2001
1997	9.5	9.5	9.6	9.7	9.7		
1998	10.0	10.0	9.9	9.8			
1999	10.0	9.8	9.7				
2000	9.5	9.5					
2001	9.0						

Table 35.2

Solution.
(a) The sequence of interest rates beginning from a given year of investment runs horizontally through the row for that year and then down the last column of rates.
An investment made at the beginning of 1992 would earn the investment year rate of 8.25% in 1992, the money rate of 8.25% in 1993, 8.4% in 1994, 8.5% in 1995 and 1996. Starting in 1997, an investment made at the beginning of 1992 would earn the portfolio average rates. The 1992 investment would earn the portfolio rate of 8.35% in 1997, the portfolio rate of 8.6% in 1998, and so on.
An investment made at the beginning of 2001 would earn the investment year rate of 9% in 2001-2005. Starting the year 2006, the investment made in 2001 would earn the portfolio rate of 8.35% in 2006, the portfolio rate of 8.6% in 2007, and so on.
(a) The accumulated value is $1000 \cdot (1.095)(1.095)(1.096) = 1,314.13$.
(b) The accumulated value is $Q = 1000 \cdot (1.0835)(1.086)(1.0885) = 1,280.82$.
(c) The accumulated value is $1000 \cdot (1.095)(1.1)(1.1) = 1,324.95$ ■

Example 35.2
An investment of 1000 is made on January 1996 in an investment fund crediting interest according to Table 35.2. How much interest is credited from January 1, 1999 to January 1, 2001?

Solution.
The accumulated value on January 1, 1999 is

$$1000(1.0925)(1.0935)(1.095) = 1308.14.$$

The accumulated value on January 1, 2001 is

$$1000(1.0925)(1.0935)(1.095)(1.0955)(1.096) = 1570.64.$$

Thus, the total amount of interest credited is

$$1570.64 - 1308.14 = 262.50 \blacksquare$$

Example 35.3
Using Table 35.2, find
(a) the interest rates credited in calendar year 2000 for deposits made in 1994 - 2000;
(b) the new money rates credited in the first year of investment for deposits made in 1992 - 2001.

Solution.
(a) 9.5% credited for new deposits made in 2000, 9.8% for deposits made in 1999, 9.9% for deposits made in 1998, 9.7% for deposits made in 1997, 9.6% for deposits made in 1996, 9.1% for deposits made in 1994 - 1995. Thus, the interest credited in the calendar year 2000 appear on an upwardly diagonal within the table.
(b) The new money rates credited in the first year of investment appear in the first column of the table ■

Practice Problems

Problem 35.1
Using Table 35.2, what were the interest rates credited in calendar year 2001 for deposits made in 2001 and before 2001?

Problem 35.2
If 100 is invested in the fund at the beginning of 2002, find the accumulated amount at the end of 2003 using:
(a) The investment year method
(b) The portfolio method
(c) The investment year method if the amount is withdrawn and then reinvested at the end of 2002.

y	i_1^y	i_2^y	i_3^y	i_4^y	i^{y+4}	Portfolio Year
1998	7.0	6.5	6.0	5.8	5.9	2002
1999	6.4	6.1	5.8	5.9	6.0	2003
2000	6.2	6.0	5.9	6.0	6.2	2004
2001	6.1	5.9	6.1	6.4	6.6	2005
2002	6.0	6.1	6.3	6.6		
2003	6.4	6.6	6.7			
2004	6.8	7.0				
2005	7.5					

Problem 35.3
The following table lists the interest rate credited under an investment year method of crediting interest.

y	i_1^y	i_2^y	i_3^y	i_4^y	i_5^y	i^{y+5}
1999	7.0	6.75	6.5	6.25	6.0	5.5
2000	6.0	5.5	5.25	5.1	5.0	
2001	5.0	4.8	4.6	4.3		
2002	4.0	3.75	3.5			
2003	3.0	3.2				
2004	4.0					

Becky invests $1,000 on January 1, 1999 and an additional $500 on January 1, 2003. How much money does Becky have on December 31, 2004?

Use the following chart of interest rates to answer Problems 35.4 through 35.6.

y	i_1^y	i_2^y	i_3^y	i_4^y	i^{y+4}	Calendar year of portfolio rate $y+4$
1997	0.0650	0.0625	0.0600	0.0575	0.0550	2001
1998	0.0600	0.0575	0.0550	0.0525	0.0500	2002
1999	0.0500	0.0475	0.0460	0.0450	0.0450	2003
2000	0.0450	0.0440	0.0430	0.0420	0.0410	2004
2001	0.0400	0.0390	0.0380	0.0370		
2002	0.0300	0.0300	0.0325			
2003	0.0300	0.0325				
2004	0.0300					

Problem 35.4
Jordan invests 1,000 in a fund on January 1, 1998. The fund uses the investment year method of determining interest rates.
Calculate the amount that Jordan will have at the end of 2004.

Problem 35.5
Jenna invests 1,000 in a fund on January 1, 2002. The fund uses the portfolio method of determining interest rates.
Calculate the amount that Jenna will have at the end of 2004.

Problem 35.6
James invests 1,000 into a fund at the beginning of each year from 2002 to 2004. The fund pays interest using the investment year interest rate.
Calculate the amount of money that James will have at the end of 2004.

Use the following chart of interest rates to answer Questions 35.7 through 35.10.

y	i_1^y	i_2^y	i_3^y	i^{y+3}
1995	3.7%	3.6%	3.5%	6.0%
1996	3.2%	3.1%	3.0%	5.5%
1997	2.7%	2.6%	2.5%	5.0%
1998	2.2%	2.1%	2.0%	4.5%
1999	1.7%	1.6%	1.5%	4.0%

Problem 35.7
A deposit of $100 is made at the beginning of 1997. How much interest was credited during 1998? What is the accumulated value at the end of 2002?

Problem 35.8
How much interest is credited in the calendar years 1997 through 1999 inclusive to a deposit of \$100 made at the beginning of 1995?

Problem 35.9
What were the interest rates credited in calendar year 1999 for deposits made in 1999, 1998, 1997, 1996, and 1995?

Problem 35.10
What were the new money rates credited in the first year of investment for deposits made in 1995, 1996, 1997, and so on?

Problem 35.11
The following table shows the annual effective interest rates credited to an investment fund by calendar year of investment. The investment year method applies for the first two years, after which a portfolio method is used.

y	i_1^y	i_2^y	i^{y+2}
1995	t	5.5%	4.5%
1996	6.0%	6.1%	5.0%
1997	7.0%	$t+2.5\%$	5.5%

An investment of \$100 is made at the beginning of 1995 and 1997. The total amount of interest credited by the fund during the year 1998 is \$13.81. Find t.

Problem 35.12
It is known that $1+i_t^y=(1.08+0.005t)^{1+0.01y}$ for integral $t,\ 0\le t\le 5$, and integral $y,\ 0\le y\le 10$. If \$1,000 is invested for three years beginning in year $y=5$, find the equivalent effective rate of interest.

Problem 35.13
For a certain portfolio, $i_t^y=(1.06+0.004t)^{1+0.01y}$ for $1\le t\le 5$ and $0\le y\le 10$.
(a) What is the accumulated value at the end of year 4 of a single investment of 1,000 at the beginning of year 2?
(b) What is the accumulated value at the end of year 4 of a deposit of 1,000 at the beginning of each of years 2 through 4?

Problem 35.14 ‡
The following table shows the annual effective interest rates being credited by an investment account, by calendar year of investment. The investment year method is applicable for the first 3 years, after which a portfolio rate is used.

y	i_1^y	i_2^y	i_3^y	i^{y+3}
1990	10%	10%	t%	8%
1991	12%	5%	10%	$(t-1)$%
1992	8%	$(t-2)$%	12%	6%
1993	9%	11%	6%	9%
1994	7%	7%	10%	10%

An investment of 100 is made at the beginning of years 1990, 1991, and 1992. The total amount of interest credited by the fund during the year 1993 is equal to 28.40. Calculate t.

Problem 35.15
Find $\ddot{s}_{\overline{5|}}$ from the data in Table 35.2 assuming the first payment is made in calendar year 1995.

Problem 35.16
You are given the following table of interest rates:

y	i_1^y	i_2^y	i^{y+5}	Portfolio Year
2004	9.00%	10.00%	11.00%	2006
2005	7.00%	8.00%		
2006	5.00%			

1000 is invested at the beginning of calendar years 2004, 2005, and 2006. What is the total amount of interest credited for calendar year 2006?

Problem 35.17
You are given:

y	i_1^y	i_2^y	i_3^y	i^{y+5}
2001	5.0	5.5	6.0	3.0
2002	4.5	5.0	5.5	2.5
2003	3.0	3.5	4.0	2.0
2004	4.0	X	5.0	
2005	5.0	5.5	6.0	
2006	4.5	5.0	5.5	

The accumulated value on 1/1/2007 of a deposit on 1/1/2001 is 9.24% greater than the accumulated value on 1/1/2007 of the same amount deposited on 1/1/2004. Determine X.

36 Yield Rates in Capital Budgeting

One of the use of yield rates is in capital budgeting. An investor always is faced with the need to allocate an amount of capital among various alternative investments in order to achieve the highest possible level of return on the investments. This process of making such financial decisions is referred to as **capital budgeting.** In this section we discuss briefly this concept. Moreover, we will assume that risk is non-existent in the alternative investments being compared.
The two major approaches to capital budgeting that are encountered in practice are the **yield rate method** and the **net present value method**.
In the yield rate method the investor computes the yield rate(s) for each alternative investment by solving the equation $NPV(i) = 0$. Then these rates are compared to an **interest preference rate** set by the investor. This is usually the minimum rate of return acceptable by the investor. All yield rates that are lower than the interest preference rate are rejected. The yield rates that are higher than the interest preference rateare the only one considered. They are ranked from highest to lowest and are selected in descending order until the amount of capital available for investment is exhausted.
In the net present value method or the NPV method, the investor computes $NPV(i)$ for each alternative investment, where i is the interest preference rate. Negative values of $NPV(i)$ are rejected and only the positive values are considered since the present value of returns is larger than that of the contributions. Capital is then allocated among those investments with positive $NPV(i)$ in such a manner that the total present value of returns from the investment (computed at the interest preference rate) minus the contributions to the investment is maximized.
It has been shown in Finance Theory that the NPV method usually provides better decisions than other methods when making capital investments. Consequently, it is the more popular evaluation method of capital budgeting projects.

Example 36.1
Consider the investment project given in the table below.
(a) Find the yield rate of this project.
(b) Assuming an interest preference rate of 3%, would you accept this project when using the yield rate method? the net present value method?

Solution.
(a) Solving the equation

$$\begin{aligned}NPV(i) =& -80,000 - 10,000\nu - 10,000\nu^2 - 10,000\nu^3 - 8,000\nu^4 + 28,000\nu^5\\&+ 38,000\nu 6 + 33,000\nu^7 + 23,000\nu^8 + 13,000\nu^9 + 8,000\nu^{10} = 0\end{aligned}$$

we find the two solutions $i = -1.714964 < -1$ and $i = 0.032180$. Thus, the yield rate is 3.218%.
(b) Using the yield rate method, since $3.218\% > 3\%$, the investor would accept this project

for consideration. Using the net present value method the investor would also accept it, since $NPV(0.03) = 1488.04 > 0$ ■

Year	Contributions	Returns	c_t
0	80,000	0	$-80,000$
1	10,000	0	$-10,000$
2	10,000	0	$-10,000$
3	10,000	0	$-10,000$
4	20,000	12,000	$-8,000$
5	2,000	30,000	28,000
6	2,000	40,000	38,000
7	2,000	35,000	33,000
8	2,000	25,000	23,000
9	2,000	15,000	13,000
10	0	8,000	8,000

Example 36.2
Repeat the same problem as above but with an interest preference rate of 4%.

Solution.
Using the yield rate method, since $4\% > 3.218\%$, the investor would reject this project. Using the net present value method the investor would also reject it, since $NPV(0.04) = -5122.13 < 0$ ■

Practice Problems

Problem 36.1
An investment project has the following cash flows:

Year	Contributions	Returns
0	100	0
1	200	0
2	10	60
3	10	80
4	10	100
5	5	120
6	0	60

(a) Using the net present value method with an interest preference rate of 15%, should the investment be accepted or rejected?
(b) Answer the same question when using the yield rate method.

Problem 36.2
An investor enters into an agreement to contribute \$7,000 immediately and \$1,000 at the end of two years in exchange for the receipt of \$4,000 at the end of one year and \$5,500 at the end of three years.
(a) Calculate $NPV(0.09)$ and $NPV(0.10)$.
(b) If an investor's interest preference rate is 12%, should the investment be accepted or rejected?

Problem 36.3
A used car can be purchased for \$5000 cash or for \$2400 down and \$1500 at the end of each of the next two years. Should a purchaser with an interest preference rate of 10% pay cash or finance the car?

Problem 36.4
Consider an investment in which a person makes payments of \$100 immediately and \$132 at the end of one year. What method would be better to use, the yield rate method or the net present value method if the preference interest rate is 15%?

Problem 36.5
Consider an investment in which a person makes payments of \$100 immediately and \$101 at the end of two years in exchange for a payment of \$200 at the end of one year. Explain why the yield rate method is not applicable in this case.

Problem 36.6
A borrower needs \$800. The funds can be obtained in two ways:
(i) By promising to pay \$900 at the end of the period.
(ii) By borrowing \$1000 and repaying \$1120 at the end of the period.
If the interest preference rate for the period is 10%, which option should be chosen?

Loan Repayment Methods

Loan is an arrangement in which a lender gives money or property to a borrower, and the borrower agrees to return the property or repay the money, usually along with interest, at some future point(s) in time.
Various methods of repaying a loan are possible. We will consider two of them: The amortization method and the sinking fund method.
The amortization method: In this method the borrower makes installment payments to the lender. Usually these payments are at a regularly spaced periodic intervals; the progressive reduction of the amount owed is described as the amortization of the loan. Examples include car loan, home mortgage repayment.
The sinking fund method: In this method the loan will be repaid by a single lump sum payment at the end of the term of the loan. The borrower pays interest on the loan in installments over this period. However, the borrower may prepare himself for the repayment by making deposits to a fund called a **sinking fund** to accumulate the repayment amount.

37 Finding the Loan Balance Using Prospective and Retrospective Methods.

When using the amortization method, the payments form an annuity whose present value is equal to the original amount of the loan. In this section, we want to determine the unpaid balance, also referred to as the **outstanding loan balance** or **unpaid principal** at any time after the inception of the loan.

There are two approaches used in finding the amount of the outstanding balance: the prospective and the retrospective method.

With the **prospective method,** the outstanding loan balance at any point in time is equal to the present value at that date of the remaining payments.

With the **retrospective method,** the outstanding loan balance at any point in time is equal to the original amount of the loan accumulated to that date less the accumulated value at that date of all payments previously made.

In general, the two approaches are equivalent. At the time of inception of the loan we have the following equality

$$\text{Present Value of All Payments} = \text{Amount of Loan}$$

Accumulate each side of the equation to the date at which the outstanding loan balance is desired, obtaining

$$\text{Current Value of Payments} = \text{Accumulated Value of Loan Amount}$$

But payments can be divided into past and future payments giving

$$\begin{gathered}\text{Accumulated Value of Past Payments} + \text{Present Value of Future Payments}\\ = \text{Accumulated Value of Loan Amount}\end{gathered}$$

or

$$\begin{gathered}\text{Present Value of Future Payments} = \text{Accumulated Value of Loan Amount}\\ - \text{Accumulated Value of Past Payments.}\end{gathered}$$

But the left side of this equation represents the prospective approach and the right side represents the retrospective approach.

We can prove that the two methods are equivalent algebraically as follows. Let B_t^p and B_t^r denote the outstanding loan balances at time t using the prospective and retrospective methods respectively. We denote the initial loan by L.

Suppose we want to repay the loan by level payments of P for n periods at a periodic interest of i.. Then P satifies the equation

$$P = \frac{L}{a_{\overline{n}|}}.$$

For $0 < t < n$, the outstanding loan balance at time t computed after making the t^{th} payment is

$$B_t^p = Pa_{\overline{n-t}|}$$

by the prospective method and

$$B_t^r = L(1+i)^t - Ps_{\overline{t}|}$$

by the retrospective method. We will show that $B_t^r = B_t^p$. Indeed,

$$\begin{aligned} B_t^r =& L(1+i)^t - Ps_{\overline{t}|} \\ =& Pa_{\overline{n}|}(1+i)^t - Ps_{\overline{t}|} \\ =& P\left[\frac{(1+i)^t - (1+i)^{-(n-t)} - (1+i)^t + 1}{i}\right] \\ =& P\frac{1-\nu^{n-t}}{i} = Pa_{\overline{n-t}|} = B_t^p. \end{aligned}$$

Example 37.1 ‡
A loan is being repaid with 25 annual payments of 300 each. With the 10th payment, the borrower pays an extra 1,000, and then repays the balance over 10 years with a revised annual payment. The effective rate of interest is 8%. Calculate the amount of the revised annual payment.

Solution.
The balance after 10 years, prospectively, is

$$B_{10}^p = 300a_{\overline{15}|} = \$2,567.84.$$

If the borrower pays an additional \$1,000, the balance becomes \$1,567.84. An equation of value for the revised payment, denoted by R, is

$$Ra_{\overline{10}|} = 1,567.84$$

or

$$R = \frac{1,567.84}{a_{\overline{10}|}} = \$233.66 \blacksquare$$

Example 37.2
A loan is being repaid with 16 quarterly payments, where the first 8 payments are each \$200 and the last 8 payments are each \$400. If the nominal rate of interest convertible quarterly is 10%, use both the prospective method and the retrospective method to find the outstanding loan balance immediately after the first six payments are made.

Solution.
With the prospective method, we have

$$\begin{aligned} B_6^p =& 200(\nu+\nu^2)+400\nu^2(\nu+\nu^2+\cdots+\nu^8)\\ =&400(\nu+\nu^2+\cdots+\nu^{10})-200(\nu+\nu^2)\\ =&400a_{\overline{10}|}-200a_{\overline{2}|}\\ =&400(8.7521)-200(1.9274)=\$3,115.36. \end{aligned}$$

With the retrospective method, we have that the original loan amount is

$$\begin{aligned} L =& 200(\nu+\nu^2+\cdots+\nu^8)+400\nu^8(\nu+\nu^2+\cdots+\nu^8)\\ =&400(\nu+\nu^2+\cdots+\nu^{16})-200(\nu+\nu^2+\cdots+\nu^8)\\ =&400a_{\overline{16}|}-200a_{\overline{8}|}\\ =&400(13.0550)-200(7.1701)=\$3,787.98 \end{aligned}$$

The outstanding loan balance is

$$B_6^r=3787.98(1.025)^6-200s_{\overline{6}|}=4392.90-200(6.3877)=\$3,115.36 \blacksquare$$

Example 37.3
Megan is buying a car for 30,000 using a 60-month car loan with an interest rate of 9% compounded monthly. For the first two years, Megan makes the required payment. Beginning with the first payment in the third year, Megan begins paying twice the required payment. Megan will completely pay off her loan by making a smaller final payment.
Determine the total number of payments that Megan will make.

Solution.
The monthly effective interest rate is $\frac{0.09}{12}=0.0075$. The original required payment is P such that $30000=Pa_{\overline{60}|0.0075}=P\frac{(1-1.0075^{-60})}{0.0075}\rightarrow P=\frac{30000\times 0.0075}{(1-1.0075^{-60})}=622.7506568.$
At the end of the first 2 years, Megan's outstanding loan balance, prospectively, is

$$622.7506568a_{\overline{36}|0.0075}=622.7506568\frac{(1-1.0075^{-36})}{0.0075}=19583.51862$$

From this point on, Megan pays $2P = 1245.501314$ every month. Then $19583.51862 = 1245.501314a_{\overline{n}|0.0075} = 1245.501314\frac{(1-1.0075^{-n})}{0.0075}$, where n is the number of months until the loan is repaid. Thus, $0.1179255197 = (1-1.0075^{-n}) \rightarrow 1.0075^{-n} = 0.8820744803 \rightarrow n = -\frac{\ln 0.8820744803}{\ln 1.0075} = 16.79315649$. So Megan will make 16 full payments and 1 smaller payment for a total of 17 additional payments and $17 + 24 = 41$ total payments over the course of repaying the loan ■

Example 37.4
A loan of 1,000 is being repaid with quarterly payments at the end of each quarter for 5 years, at 6% convertible quarterly. Find the outstanding loan balance at the end of the 2nd year.

Solution.
The quarterly effective interest rate is $\frac{0.06}{4} = 0.015$. Let P be the quarterly payment. Then $1000 = Pa_{\overline{20}|0.015} = P\frac{(1-1.015^{-20})}{0.015} \rightarrow P = \frac{1000\times 0.015}{(1-1.015^{-20})} = 58.24573587$. The outstanding loan balance at the end of the 2nd year, prospectively, is $58.24573587a_{\overline{12}|0.015} = 58.24573587\frac{(1-1.015^{-12})}{0.015} = \$635.32$ ■

Practice Problems

Problem 37.1
A loan is being repaid with level annual payments of \$1,000. Calculate the outstanding balance of the loan if there are 12 payments left. The next payment will be paid one year from now and the effective annual interest rate is 5%.

Problem 37.2
A loan of 10,000 is being repaid will 20 non-level annual payments. The interest rate on the loan is an annual effective rate of 6%. The loan was originated 4 years ago. Payments of 500 at the end of the first year, 750 at the end of the second year, 1,000 at the end of the third year and 1,250 at the end of the fourth year have been paid.
Calculate the outstanding balance immediately after the fourth payment.

Problem 37.3
Calculate the outstanding balance to the loan in the previous problem one year after the fourth payment immediately before the fifth payment.

Problem 37.4
Julie bought a house with a 100,000 mortgage for 30 years being repaid with payments at the end of each month at an interest rate of 8% compounded monthly. What is the outstanding balance at the end of 10 years immediately after the 120th payment?

Problem 37.5
If Julie pays an extra 100 each month, what is the outstanding balance at the end of 10 years immediately after the 120th payment?

Problem 37.6
A loan 20,000 is being repaid with annual payments of 2,000 at the end of each year. The interest rate charged on the loan is an annual effective rate of 8%. Calculate the outstanding balance of the loan immediately after the 5th payment.

Problem 37.7
The interest rate on a 30 year mortgage is 12% compounded monthly. The mortgage is being repaid by monthly payments of 700. Calculate the outstanding balance at the end of 10 years.

Problem 37.8
The interest rate on a 30 year mortgage is 12% compounded monthly. Lauren is repaying the mortgage by paying monthly payments of 700. Additionally, to pay off the loan early, Lauren has made additional payments of 1,000 at the end of each year. Calculate the outstanding balance at the end of 10 years.

Problem 37.9
A loan is being repaid with 20 payments of 1,000. The total interest paid during the life of the loan is 5,000. Calculate the amount of the loan.

Problem 37.10
A loan of 10,000 is being repaid by installments of 2,000 at the end of each year, and a smaller final payment made one year after the last regular payment. Interest is at the effective rate of 12%. Find the amount of outstanding loan balance remaining when the borrower has made payments equal to the amount of the loan.

Problem 37.11
A loan is being repaid by quarterly installments of 1,500 at the end of each quarter, at 10% convertible quarterly. If the loan balance at the end of the first year is 12,000, find the original loan balance.

Problem 37.12
A loan is being repaid by 15 annual payments at the end of each year. The first 5 installments are 4,000 each, the next 5 are 3,000 each, and the final 5 are 2,000 each. Find expressions for the outstanding loan balance immediately after the second 3,000 installment.
(a) prospectively;
(b) retrospectively.

Problem 37.13
A loan is to be repaid with level installments payable at the end of each half-year for $3\frac{1}{2}$ years, at a nominal rate of interest of 8% convertible semiannually. After the fourth payment the outstanding loan balance is 5,000. Find the initial amount of the loan.

Problem 37.14
A 20,000 loan is to be repaid with annual payments at the end of each year for 12 years. If $(1+i)^4 = 2$, find the outstanding balance immediately after the fourth payment.

Problem 37.15 ‡
A 20-year loan of 1,000 is repaid with payments at the end of each year. Each of the first ten payments equals 150% of the amount of interest due. Each of the last ten payments is X. The lender charges interest at an annual effective rate of 10%. Calculate X.

Problem 37.16 ‡
A loan is amortized over five years with monthly payments at a nominal interest rate of 9% compounded monthly. The first payment is 1,000 and is to be paid one month from the date of the loan. Each succeeding monthly payment will be 2% lower than the prior payment. Calculate the outstanding loan balance immediately after the 40th payment is made.

Problem 37.17
A mortgage loan is being repaid with level annual payments of 5,000 at the end of the year for 20 years. The interest rate on the mortgage is 10% per year. The borrower pays 10 payments and then is unable to make payments for two years. Calculate the outstanding balance at the end of the 12th year.

Problem 37.18 ‡
An investor took out a loan of 150,000 at 8% compounded quarterly, to be repaid over 10 years with quarterly payments of 5483.36 at the end of each quarter. After 12 payments, the interest rate dropped to 6% compounded quarterly. The new quarterly payment dropped to 5134.62.
After 20 payments in total, the interest rate on the loan increased to 7% compounded quarterly. The investor decided to make an additional payment of X at the time of his 20th payment. After the additional payment was made, the new quarterly payment was calculated to be 4265.73, payable for five more years. Determine X.

Problem 37.19 ‡
A small business takes out a loan of 12,000 at a nominal rate of 12%, compounded quarterly, to help finance its start-up costs. Payments of 750 are made at the end of every 6 months for as long as is necessary to pay back the loan.
Three months before the 9th payment is due, the company refinances the loan at a nominal rate of 9%, compounded monthly. Under the refinanced loan, payments of R are to be made monthly, with the first monthly payment to be made at the same time that the 9th payment under the old loan was to be made. A total of 30 monthly payments will completely pay off the loan. Determine R.

Problem 37.20
A loan of $1,000 is being repaid with quarterly payments at the end of each quarter for 5 years, at 6% convertible quarterly. Find the outstanding loan balance at the end of the second year.

Problem 37.21
A 20,000 mortgage is being repaid with 20 annual installments at the end of each year. The borrower makes 5 payments, and then is temporarily unable to make payments for the next 2 years. Find an expression for the revised payment to start at the end of the 8th year if the loan is still to be repaid at the end of the original 20 years.

Problem 37.22
A loan of 1 was originally scheduled to be repaid by 25 equal annual payments at the end of each

year. An extra payment K with each of the 6th through the 10th scheduled payments will be sufficient to repay the loan 5 years earlier than under the original schedule. Show that

$$K = \frac{a_{\overline{20}|} - a_{\overline{15}|}}{a_{\overline{25}|}a_{\overline{5}|}}.$$

Problem 37.23
Bob takes out a loan of 1000 at an annual effective interest rate of i. You are given:
(a) The first payment is made at the end of year 6.
(b) Ten equal annual payments are made to repay the loan in full at the end of 15 years.
(c) The outstanding principal after the payment made at the end of year 10 is 908.91.
Calculate the outstanding principal at the end of year 5.

Problem 37.24
The original amount of an inheritance was just sufficient at 3.5% effective to pay \$10,000 at the end of each year for 10 years. The payments were made as scheduled for the first five years even though the fund actually earned 5% effective. How much excess interest was in the fund at the end of the fifth year?

38 Amortization Schedules

When a loan is being repaid with the amortization method, each payment is partially a repayment of principal and partially a payment of interest. Determining the amount of each for a payment can be important (for income tax purposes, for example).
An **amortization schedule** is a table which shows the division of each payment into principal and interest, together with the outstanding loan balance after each payment is made.
Consider a loan of $a_{\overline{n}|}$ at interest rate i per period being repaid with payments of 1 at the end of each period for n periods. At the end of period 1 (i.e. after the first payment), the interest paid is $ia_{\overline{n}|} = 1-\nu^n$ so that the principal repaid is ν^n, and the outstanding loan balance is $a_{\overline{n}|} - \nu^n = a_{\overline{n-1}|}$. Next, at the end of the second period, the interest paid is $ia_{\overline{n-1}|} = 1-\nu^{n-1}$ so that the principal repaid is ν^{n-1}, and the outstanding loan balance is $a_{\overline{n-1}|} - \nu^{n-1} = a_{\overline{n-2}|}$. Continuing this process, we see that at the end of period k, the interest paid is $ia_{\overline{n-k+1}|} = 1-\nu^{n-k+1}$ and the principal repaid is ν^{n-k+1}. The outstanding loan balance is $a_{\overline{n-k+1}|} - \nu^{n-k+1} = a_{\overline{n-k}|} = B_k^p$. The amortization table is shown below.

Period	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				$a_{\overline{n}|}$
1	1	$ia_{\overline{n}|} = 1-\nu^n$	ν^n	$a_{\overline{n}|} - \nu^n = a_{\overline{n-1}|}$
2	1	$ia_{\overline{n-1}|} = 1-\nu^{n-1}$	ν^{n-1}	$a_{\overline{n-1}|} - \nu^{n-1} = a_{\overline{n-2}|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	1	$ia_{\overline{n-k+1}|} = 1-\nu^{n-k+1}$	ν^{n-k+1}	$a_{\overline{n-k+1}|} - \nu^{n-k+1} = a_{\overline{n-k}|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	1	$ia_{\overline{2}|} = 1-\nu^2$	ν^2	$a_{\overline{2}|} - \nu^2 = a_{\overline{1}|}$
n	1	$ia_{\overline{1}|} = 1-\nu$	ν	$a_{\overline{1}|} - \nu = 0$
Total	n	$n - a_{\overline{n}|}$	$a_{\overline{n}|}$	

Observe each of the following from this table: First, it should be noted that the outstanding loan balance agrees with that obtained by the prospective method. Second, the sum of the principal repayments equals to the original amount of the loan. Third, the sum of interest payments is equal to the difference between the sum of the total payments and the sum of the principal repayments. Fourth, the sum of principal repayments is a geometric progression with common ration $(1+i)$.

Example 38.1
Create an amortization schedule for a loan of \$1,000 repaid over four years if the annual effective rate of interest is 8%.

Solution.
If R is the periodic payment then

$$R = \frac{1000}{a_{\overline{4}|}} = \$301.92.$$

Period	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				1000.00
1	301.92	80.00	221.92	778.08
2	301.92	62.25	239.67	538.41
3	301.92	43.07	258.85	279.56
4	301.92	22.36	279.56	0

Table 38.1

In the previous example, the last line exactly balances. That is, after the last payment is made the loan is paid off. In practice, there will be rounding errors as the table is generated line by line, and the last line may not lead to a zero balance. Standard practice is to adjust the last payment so that it is exactly equal to the amount of interest for the final period plus the outstanding loan balance at the beginning of the final period, in order to bring the outstanding loan balance to 0.

Example 38.2
Create an amortization schedule for a loan of \$10,000 repaid over three years if the annual effective rate of interest is 7%.

Solution.
If R is the periodic payment then

$$R = \frac{10,000}{a_{\overline{3}|}} = \$3,810.52.$$

Period	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				10,000.00
1	3,810.52	700.00	3,110.52	6,889.48
2	3,810.52	482.26	3,328.26	3,561.22
3	3,810.50	249.28	3,561.22	0

It should be noted that if it is desired to find the amount of principal and interest in one particular payment, it is not necessary to construct the entire amortization schedule. The outstanding loan balance at the beginning of the period in question can be determined by either the retrospective or the prospective methods, and then that one line of the schedule can be created.

Example 38.3
Jones borrows \$20,000 from Smith and agrees to repay the loan with equal quarterly installments of principal and interest at 10% convertible quarterly over eight years. At the end of three years, Smith sells the right to receive future payments to Collins at a price which produces a yield rate of 12% convertible quarterly for Smith. Find the total amount of interest received (a) by Collins, and (b) by Smith.

Solution.
(a) Each quarterly payment by Jones is

$$\frac{20000}{a_{\overline{32}|0.025}} = \$915.37.$$

Total payments by Jones to Collins over the last five years are

$$(20)915.37 = \$18307.40.$$

After three years, the price Collins pays to Smith is

$$915.37a_{\overline{20}|0.03} = 915.37(14.8775) = \$13618.42.$$

Total amount of interest received by Collins is

$$18307.40 - 13618.42 = \$4688.98.$$

(b) Total payments by Jones to Smith over the first three years are

$$(12)915.37 = \$10984.44.$$

After three years, the outstanding loan balance is

$$915.37a_{\overline{20}|0.025} = 915.37(15.5892) = \$14269.89.$$

Recall from (a) that the price Collins pays to Smith after three years is \$13618.42. Total amount of interest received by Smith is

$$13618.42 + 10984.44 - 20000 = \$4602.86 \blacksquare$$

Let I_t denote the amount of interest paid in the t^{th} installment; P_t be the amount of principal and B_t be the loan balance.

Example 38.4
Show that for $n = 100$ and with a periodic payment of R, we have

$$P_{11} + P_{12} + \cdots + P_{50} = B_{10} - B_{50}$$

and

$$I_{11} + I_{12} + \cdots + I_{50} = 40R - (B_{10} - B_{50}).$$

Solution.
Let $P = P_{11} + P_{12} + \cdots + P_{50}$. Then

$$\begin{aligned}
P =& R\sum_{k=11}^{50} \nu^{100-k+1} = R\nu^{101}\sum_{k=11}^{50}(1+i)^k \\
=& Rv^{101}(\sum_{k=0}^{50}(1+i)^k - \sum_{k=0}^{10}(1+i)^k) \\
=& R\frac{(1+i)^{-50} - (1+i)^{-90}}{i} \\
=& Ra_{\overline{90}|} - Ra_{\overline{50}|} = B_{10} - B_{50}
\end{aligned}$$

Now, let $I = I_{11} + I_{12} + \cdots + I_{50}$ then $I = \sum_{j=11}^{50} R - \sum_{j=11}^{50} P_j = 40R - (B_{10} - B_{50})$ ■

Amortization schedules of perpetuities do not exist since the entire payment represents interest and therefore the loan balance remains unchanged.

Example 38.5
A \$5,000 loan is being repaid by payments of \$X at the end of each half year for as long as necessary until a smaller final payment is made. The nominal rate of interest convertible semiannually is 14%.
(a) If $X = \$400$ find the amount of principal and the interest in the sixth payment.
(b) If $X = \$350$, find the principal in the sixth payment, and interpret this.

Solution.
(a) The outstanding balance at the beginning of the sixth half-year is

$$B_5^r = 5000(1.07)^5 - 400s_{\overline{5}|} = 7012.76 - 400(5.7507) = \$4712.48.$$

The interest in the sixth payment is $(0.07)(4712.48) = \$329.87$ and the principal is $400 - 329.87 = \$70.13$.
(b) The outstanding balance at the beginning of the sixth half-year is

$$B_5^r = 5000(1.07)^5 - 350s_{\overline{5}|} = 7012.76 - 350(5.7507) = \$5000.01.$$

If $X \leq 350$, then the loan will never be paid off, because every payment will count only toward interest ∎

Thus far, in creating an amortization schedule we have assumed a constant rate of interest, the conversion period and payment period are the same, and the payments are leveled. Example 38.2 is an example where the payments are not all leveled. It is possible to create an amortization schedule with varying interest rate as illustrated in the next example.

Example 38.6
An amount is invested at an annual effective rate of interest i which is exactly sufficient to pay 1 at the end of each year for n years. In the first year, the fund earns rate i and 1 is paid at the end of the year. However, in the second year, the fund earns rate $j > i$. If X is the revised payment which could be made at the end of years 2 to n, then find X assuming that
(a) the rate reverts back to i again after this one year,
(b) the rate earned remains at j for the rest of the $n-$year period.

Solution.
(a) From the amortization table, the balance at the end of the first year is $a_{\overline{n-1}|i}$, and therefore the balance after two years must be $(1+j)a_{\overline{n-1}|i} - X$. However, after two years, the balance must be equal to the present value of all future payments. Consequently, we have that

$$\begin{aligned}
(1+j)a_{\overline{n-1}|i} - X =& Xa_{\overline{n-2}|i}\\
X(1+a_{\overline{n-2}|i}) =& (1+j)a_{\overline{n-1}|i}\\
X(1+i)a_{\overline{n-1}|i} =& (1+j)a_{\overline{n-1}|i}\\
X =& \frac{1+j}{1+i}.
\end{aligned}$$

(b) As in (a), the balance at the end of two years is $(1+j)a_{\overline{n-1}|i} - X$. However, after two years, the balance must be equal to the present value of all future payments. Consequently, we have that

$$\begin{aligned}
(1+j)a_{\overline{n-1}|i} - X =& Xa_{\overline{n-2}|j}\\
X(1+a_{\overline{n-2}|j}) =& (1+j)a_{\overline{n-1}|i}\\
X(1+j)a_{\overline{n-1}|j} =& (1+j)a_{\overline{n-1}|i}\\
X =& \frac{a_{\overline{n-1}|i}}{a_{\overline{n-1}|j}}
\end{aligned}$$ ∎

Situations where payment period and interest conversion period do not coincide are best handled

from first principles rather than from developing formulas.
With an amortization schedule in which payments are made at a different frequency than interest is convertible, the following two-step procedure can be followed:
(1) Find the rate of interest which is equivalent to the given rate of interest and convertible at the same frequency as payments are made.
(2) Construct the amortization schedule with the rate of interest in step 1.

Example 38.7
A 25-year mortgage for $100,000 bears an interest rate of 5.5% compounded semi-annually. The payments are made at the end of each month. Find the size of the payment.

Solution.
The monthly interest rate is $j = (1.0275)^{\frac{1}{6}} - 1 = .00453168$. The size of each payment is

$$R = \frac{100,000}{a_{\overline{300}|j}} = 610.39 \blacksquare$$

Example 38.8
A loan of $3,000 is being amortized by 20 quarterly payments. Payments 11 and 12 are not made. At the designated time of the 12th payment, the loan is renegotiated so that the 13th payment is $N and payments 14, 16, 18, and 20 are each $40 more that the preceding payment. If the rate of interest is 8% convertible semiannually, find the value of N which would provide that the loan be completely repaid after 20 quarters.

Solution.
The quarterly effective rate of interest is $j = (1.02)^{\frac{1}{2}} - 1$. The original quareterly payment is$R = \frac{3000}{a_{\overline{20}|j}}$.
The outstanding balance at time $t = 10$, by the prospective method, is $Ra_{\overline{10}|j}$. The outstanding balance at time $t = 12$ is

$$Ra_{\overline{10}|j}(1+j)^2 = 3000\frac{a_{\overline{10}|j}}{a_{\overline{20}|j}}(1+j)^2 = 1712.46.$$

Now, the new payments stream starting at time $t = 13$ are:

$$N, N+40, N+40, N+80, N+80, N+120, N+120, N+160.$$

By the prospective method

$$1712.46 = Na_{\overline{8}|j} + 40(\nu_j^2 + \nu_j^3 + 2\nu_j^4 + 2\nu_j^5 + 3\nu_j^6 + 3\nu_j^7 + 4\nu_j^8).$$

Solving this equation for N we find $N = 155.73 \blacksquare$

Practice Problems

Problem 38.1
A loan of 10,000 is being repaid with annual payments of 1500 for 11 years. Calculate the amount of principal paid over the life of the loan.

Problem 38.2
A loan of 10000 is being repaid with annual payments of 1500 for 11 years. Calculate the amount of interest paid over the life of the loan.

Problem 38.3
A loan is being repaid with level annual payments based on an annual effective interest rate of 8%. If the amount of principal in the 10th payment is 100, calculate the amount of principal in the 5th payment.

Problem 38.4
A loan is being repaid with level annual payments based on an annual effective interest rate of 8%. If the outstanding balance immediately after the 10th payment is 1000, calculate the amount of interest in the 11th payment.

Problem 38.5
A loan of 10,000 is being repaid with annual payments of 1500 for n years. The total principal paid in the first payment is 685.58. Calculate the interest rate on the loan.

Problem 38.6
For a loan with level annual payments, the principal repaid by the 10th payment is 10,000 while the principle repaid by the 11th payment is 11,000. Calculate the principal repaid by the 15th payment.

Problem 38.7
A 60-month loan is to be repaid with level payments of 1000 at the end of each month. The interest in the last payment is 7.44. Calculate the total interest paid over the life of the loan.

Problem 38.8
A 60-month loan is to be repaid with level payments of 1000 at the end of each month. The principal in the first payment is 671.21. Calculate the effective annual interest rate.

Problem 38.9
Jenna is repaying a 120-month loan with interest compounded monthly at 12%. Calculate the payment in which the absolute value of the difference between the interest paid and the principal repaid is minimized.

Problem 38.10
A loan is being repaid with level payments at the end of each year for 20 years. The principal repaid in the 10th payment is 1000 and the principal repaid in the 15th payment is 1200. Calculate the amount of the loan.

Problem 38.11
A loan of 10,000 is being repaid with annual payments over 10 years at 8% compound monthly. Create the amortization table for this loan.

Problem 38.12
A loan is being repaid with quarterly installments of 1,000 at the end of each quarter for 5 years at 12% convertible quarterly. Find the amount of principal in the 6th installment.

Problem 38.13
Consider a loan which is being repaid with installments of 1 at the end of each period for n periods. Find an expression at issue for the present value of the interest which will be paid over the life of the loan.

Problem 38.14
A loan of 10,000 is being repaid with 20 installments at the end of each year, at 10% effective. Show that the amount of interest in the 11th installment is

$$\frac{1000}{1+\nu^{10}}.$$

Problem 38.15
A loan is being repaid with 20 installments at the end of each year at 9% effective. In what installment are the principal and interest portions most nearly equal to each other?

Problem 38.16
A loan is being repaid with a series of payments at the end of each quarter, for 5 years. If the amount of principal in the 3rd payment is 100, find the amount of principal in the last 5 payments. Interest is at the rate of 10% convertible quarterly.

Problem 38.17
A loan is being repaid with installments of 1 at the end of each year for 20 years. Interest is at effective rate i for the first 10 years, and effective rate j for the second 10 years. Find expressions for
(a) the amount of interest paid in the 5th installment;
(b) the amount of principal repaid in the 15th installment.

Problem 38.18
A loan of 25,000 is being repaid with annual payments of 2,243 at the end of the year. The interest rate on the loan 7.5%. Calculate the interest in the 5th payment.

Problem 38.19 ‡
Seth borrows X for four years at an annual effective interest rate of 8%, to be repaid with equal payments at the end of each year. The outstanding loan balance at the end of the second year is 1076.82 and at the end of the third year is 559.12 . Calculate the principal repaid in the first payment.

Problem 38.20 ‡
A bank customer takes out a loan of 500 with a 16% nominal interest rate convertible quarterly. The customer makes payments of 20 at the end of each quarter. Calculate the amount of principal in the fourth payment.

Problem 38.21 ‡
A loan is repaid with level annual payments based on an annual effective interest rate of 7%. The 8th payment consists of 789 of interest and 211 of principal. Calculate the amount of interest paid in the 18th payment.

Problem 38.22 ‡
Kevin takes out a 10-year loan of L, which he repays by the amortization method at an annual effective interest rate of i. Kevin makes payments of 1000 at the end of each year . The total amount of interest repaid during the life of the loan is also equal to L. Calculate the amount of interest repaid during the first year of the loan.

Problem 38.23 ‡
Ron is repaying a loan with payments of 1 at the end of each year for n years. The amount of interest paid in period t plus the amount of principal repaid in period $t+1$ equals X. Calculate X.

Problem 38.24
A 35-year loan is to be repaid with equal installments at the end of each year. The amount of interest paid in the 8th installment is 135. The amount of interest paid in the 22nd installment is 108. Calculate the amount of interest paid in the 29th installment.

Problem 38.25
A 1000 loan is repaid with equal payments at the end of each year for 20 years. The principal portion of the 13th payment is 1.5 times the principal portion of the 5th payment. Calculate the total amount of interest paid on the loan.

Problem 38.26
A loan is being amortized with payments at the end of each quarter for 25 years. If the amount of principal repaid in the third payment is \$100, find the total amount of principal repaid in the forty payments consisting of payments eleven through fifty. Interest is at the rate of 8% convertible quarterly.

Problem 38.27
A loan is being amortized using an interest rate of 8% convertible quarterly with level payments at the end of each quarter. If the amount of principal repaid in the third payment is \$100, find the total amount of principal repaid in the five payments consisting of payments eleven through fifteen.

Problem 38.28
A 100,000 loan is to be repaid by 30 equal payments at the end of each year. The outstanding balance is amortized at 4%. In addition to the annual payments, the borrower must pay an origination fee at the time the loan is made. The fee is 2% of the loan but does not reduce the loan balance. When the second payment is due, the borrower pays the remaining loan balance. Determine the yield to the lender considering the origination fee and the early pay-off of the loan.

Problem 38.29
A loan of 10,000 is amortized by equal annual payments for 30 years at an annual effective interest rate of 5%. Determine the year in which the interest portion of the payment is most nearly equal to one-third of the payment.

Problem 38.30
A loan of 1000 at a nominal rate of 12% convertible monthly is to be repaid by six monthly payments with the first payment due at the end of 1 month. The first three payments are X each, and the final three payments are $3X$ each. Determine the sum of the principal repaid in the third payment and the interest paid in the fifth payment.

Problem 38.31
Carl borrows \$10,000 at 12% compounded monthly, and Carl will repay the loan with 60 monthly amortization payments beginning at the end of the first month. Find:
(a) B_{20} (b) B_{30} (c) $\sum_{j=21}^{30} P_j$ (d) $\sum_{j=21}^{30} I_j$

Problem 38.32 ‡
A bank customer borrows X at an annual effective rate of 12.5% and makes level payments at the end of each year for n years. The interest portion of the final payment is 153.86. The total principal repaid as of time $(n-1)$ is 6009.12. The principal repaid in the first payment is Y. Calculate Y.

Problem 38.33
A loan is to be amortized by n level annual payments of X where $n > 5$. You are given
(1) The amount of interest in the first payment is 604.00
(2) The amount of interest in the third payment is 593.75
(3) The amount of interest in the fifth payment is 582.45
Calculate X.

Problem 38.34
A loan is to be repaid by annual installments of X at the end of each year for 10 years. You are given
(1) the total principal repaid in the first 3 years is 290.35
(2) the total principal repaid in the last 3 years is 408.55
Calculate the total amount of interest paid during the life of the loan.

Problem 38.35
Iggy borrows X for 10 years at an annual effective rate of 6%. If he pays the principal and accumulated interest in one lump sum at the end of 10 years, he would pay 356.54 more in interest than if he repaid the loan with 10 level payments at the end of each year. Calculate X.

Problem 38.36
A loan is being amortized by means of level monthly payments at an annual effective interest rate of 8%. The amount of principal repaid in the 12th payment is 1000 and the amount of principal repaid in the t^{th} payment is 3700. Calculate t.

Problem 38.37
John is repaying a loan with payments of \$3,000 at the end of every year over an unknown period of time. if the amount on interest in the third installment is \$2,000, find the amount of principal in the sixth installment. Assume the interest is 10% convertible quarterly.

39 Sinking Fund Method

An alternative for repaying a loan in installments by the amortization method, a borrower can accumulate a fund which will exactly repay the loan in one lump sum at the end of a specified period of time. This fund is called a **sinking fund.** It is generally required that the borrower periodically pay interest on the loan, sometimes referred to as a **service.**
In some cases payments into a sinking fund may vary irregularly at the discretion of the borrower. However, we will be dealing with sinking funds with regular contributions.
Because the balance in the sinking fund could be applied against the loan at any point, the net amount of the loan is equal to the original amount of the loan minus the sinking fund balance.
We next show that if the rate of interest paid on the loan equals the rate of interest earned on the sinking fund, then the amortization method and the sinking fund method are equivalent.
To see this, Suppose that the effective annual rate on the loan is i, and the sinking fund earns the same rate. Suppose that the amount of the loan is 1 , and the loan term is n periods. With the amortization method, the payment at the end of each period is $\frac{1}{a_{\overline{n}|}}$. With the sinking fund method, to accumulate the amount of 1 in the sinking fund, the borrower deposits of $\frac{1}{s_{\overline{n}|}}$ at the end of each year for n years. At the same moment the borrower also pays i per period to the lender. That is, payments of size i (interest) plus payments of size $\frac{1}{s_{\overline{n}|}}$ are required. But we know from Section 15 that

$$\frac{1}{s_{\overline{n}|}} + i = \frac{1}{a_{\overline{n}|}}.$$

Thus, the two methods are equivalent.
As was seen in Section 38, a way to visualize the activity of an amortization method is the creation of an amortization schedule. The same idea can be used with sinking funds. The following example illustrates the creation of a **sinking fund schedule.**

Example 39.1
Create a sinking fund schedule for a loan of \$1000 repaid over four years if the annual effective rate of interest is 8%

Solution.
If R is the sinking fund deposit then

$$R = \frac{1000}{s_{\overline{4}|}} = \$221.92.$$

Period	Interest paid	Sinking fund deposit	Interest earned on sinking fund	Amount in sinking fund	Net amount of loan
0					1000.00
1	80.00	221.92	0	221.92	778.08
2	80.00	221.92	17.75	461.59	538.41
3	80.00	221.92	36.93	720.44	279.56
4	80.00	221.92	57.64	1000.00	0

Table 39.1

Notice from this table that the amount in the sinking fund after the t^{th} payment is found by multiplying the sinking fund deposit by $s_{\overline{t}|j}$ where j is the sinking fund rate of interest.
Comparing Table 38.1 and Table 39.1 we notice the following:
(1) For each period, Interest paid + Sinking Fund deposit (in Table 39.1) = Payment amount (in Table 38.1).
(2) For each period, Interest paid − Interest earned on sinking fund (in Table 39.1) = Interest paid (in Table 38.1).
(3) For each period, Sinking fund deposit + Interest earned on sinking fund (in Table 39.1) = principal repaid (in Table 38.1).
(4) For each period, the Net amount of loan (in Table 39.1) = Outstanding loan balance in (Table 38.1)
Notice that the net amount of loan concept plays the same role for the sinking fund method that the outsanding loan balance does for the amortization method.
Next, we consider the situation in which the interest rate on the loan and the interest rate earned on the sinking fund differs. The rate on the loan is denoted by i, and the rate on the sinking fund is denoted by j. Usually, j is less than i because the sinking fund wouldn't normally be riskier than the loan. In this case, an amount of i will be deducted from the sinking fund deposit and the remaining amount will be invested in the sinking fund at a rate of j.
Let $a_{\overline{n}|i\&j}$ be the present value of an annuity which pays 1 at the end of each period for n periods with i and j as previously defined. If a loan of 1 is made, by the amortization method the amount of the loan will be repaid by interest payments of $ia_{\overline{n}|i\&j}$ at the end of each year for n years together with yearly deposits of $1 - ia_{\overline{n}|i\&j}$ into a sinking fund which earns interest at the effective annual rate of interest j. The sinking find should accumulate $a_{\overline{n}|i\&j}$ at the end of n years. That is,

$$(1 - ia_{\overline{n}|i\&j})s_{\overline{n}|j} = a_{\overline{n}|i\&j}$$

or

$$a_{\overline{n}|i\&j} = \frac{s_{\overline{n}|j}}{1 + is_{\overline{n}|j}}$$

which is equivalent to

$$\frac{1}{a_{\overline{n}|i\&j}} = \frac{1}{s_{\overline{n}|j}} + i.$$

In other words, loan of 1 can be repaid by interest payments of i to the lender and depositing $\frac{1}{s_{\overline{n}|j}}$ into the sinking fund at the end of each year for n years.
Now since

$$\frac{1}{a_{\overline{n}|j}} = \frac{1}{s_{\overline{n}|j}} + j$$

it follows that

$$\frac{1}{a_{\overline{n}|i\&j}} = \frac{1}{a_{\overline{n}|j}} + (i-j)$$

or

$$a_{\overline{n}|i\&j} = \frac{a_{\overline{n}|j}}{1+(i-j)a_{\overline{n}|j}}.$$

It should be noted that if $i = j$, then $a_{\overline{n}|i\&j} = a_{\overline{n}|i}$.

Example 39.2 ‡
John borrows 10,000 for 10 years at an annual effective interest rate of 10%. He can repay this loan using the amortization method with payments of 1,627.45 at the end of each year. Instead, John repays the 10,000 using a sinking fund that pays an annual effective interest rate of 14%. The deposits to the sinking fund are equal to 1,627.45 minus the interest on the loan and are made at the end of each year for 10 years. Determine the balance in the sinking fund immediately after repayment of the loan.

Solution.
Under the amortization method, the periodic installment is 1627.45. The periodic interest payment on the loan is $0.1(10000) = 1000$. Thus, deposits into the sinking fund are $1627.45 - 1000 = 627.45$. Thus, the amount in sinking fund immediately after repayment of the loan is $627.45s_{\overline{10}|0.14} - 10,000 = 2130$ ■

In general, the sinking fund schedule at two rates of interest is identical to the sinking fund schedule at one rate of interest equal to the rate of interest earned on the sinking fund, except that a constant addition of $(i-j)$ times the amount of the original loan is added to the interest paid column. We illustrate this in the next example.

Example 39.3
Create a sinking fund schedule for a loan of \$1,000 repaid over four years if the annual effective rate of interest is 10% and the sinking fund interest of 8%

Solution.
If R is the sinking fund deposit then

$$R = \frac{1,000}{s_{\overline{4}|0.08}} = \$221.92.$$

Period	Interest paid	Sinking fund deposit	Interest earned on sinking fund	Amount in sinking fund	Net amount of loan
0					1000.00
1	100.00	221.92	0	221.92	778.08
2	100.00	221.92	17.75	461.59	538.41
3	100.00	221.92	36.93	720.44	279.56
4	100.00	221.92	57.64	1000	0

Note that this table is identical to Table 39.1 except each entry in the interest paid column in the table above is equal to each entry in the interest paid column in Table 39.1 increased by a constant equals to 20 which is $(0.10 - 0.08) \times 1000$ ■

The following is an important example whose results are to keep in mind.

Example 39.4
A loan of 1 yields the lender rate i per period for n periods, while the borrower replaces the capital in a sinking fund earning rate j per period. Find expressions for the following if $1 \le t \le n$:
(a) Periodic interest paid to the lender.
(b) Periodic sinking fund deposit.
(c) Interest earned on sinking fund during the tth period.
(d) Amount in sinking fund at end of the tth period.
(e) Net amount of loan at the end of the tth period.
(f) Net interest paid in period t.
(g) Principal repaid in period t.

Solution.
(a) The yield rate is i, so the lender must be receiving an amount of i each period.
(b) The capital is 1, so the borrower is depositing $(s_{\overline{n}|j})^{-1}$ regularly into the sinking fund.
(c) At the beginning of the tth period the balance in the sinking fund is $(s_{\overline{t-1}|j}/s_{\overline{n}|j})$; during the period it earns interest in the amount of $j \cdot \frac{s_{\overline{t-1}|j}}{s_{\overline{n}|j}}$ payable at the end of the period, i.e., at time t.
(d) The amount in sinking fund at end of the tth period is $\frac{s_{\overline{t}|j}}{s_{\overline{n}|j}}$.

(e) The net amount of loan at the end of the tth period is the excess of 1 over the balance in the sinking fund, i.e., $1 - \frac{s_{\overline{t}|j}}{s_{\overline{n}|j}}$.
(f) The net interest paid in the tth period is the excess of interest paid over interest earned, i.e. $i - j\frac{s_{\overline{t-1}|j}}{s_{\overline{n}|j}}$.
(g) By (e) above, the change in the amount of the loan between the $(t-1)$th and the tth payment is

$$\left(1 - \frac{s_{\overline{t}|j}}{s_{\overline{n}|j}}\right) - \left(1 - \frac{s_{\overline{t-1}|j}}{s_{\overline{n}|j}}\right) = \frac{(1+i)^{t-1}}{s_{\overline{n}|j}} \blacksquare$$

In creating the sinking fund schedule the payment period and the conversion interest period can be different. These cases can be handled from basic principles as illustrated in the next two examples. Remember to use the rate of interest which is equivalent to the given rate of interest and convertible at the same frequency as payments are made.

Example 39.5
John borrows \$5,000 for 10 years at 10% convertible quarterly. John does not pay interest currently and will pay all accrued interest at the end of 10 years together with the principal. Find the annual sinking fund deposit necessary to liquidate the loan at the end of 10 years if the sinking fund earns 7% convertible semi-annually.

Solution.
Let j be the annual interest rate. Then $j = (1.025)^4 - 1$. The loan balance at the end of 10 years is $5000(1+j)^{10} = 5000(1.025)^{40} = 13425.32$. Hence, the annual sinking fund deposit is

$$R = \frac{13425.32}{s_{\overline{10}|(1.035)^2-1}} = 966.08 \blacksquare$$

Example 39.6
Create the sinking fund table for the following: a 3 year loan of \$10,000, with interest payable semiannually at the nominal interest rate of 8.00% is to be retired by a sinking fund funded by quarterly deposits earning an effective nominal interest rate of 6.00% compounded semiannually.

Solution.
Every six months the interest paid on the loan is $0.04(10,000) = 400$. The quarterly rate on the sinking fund is $j = 1.03^{0.5} - 1$. The quarterly sinking fund deposit is

$$D = \frac{10,000}{s_{\overline{12}|j}} = 767.28.$$

Period	Interest paid	Sinking fund deposit	Interest earned on sinking fund	Amount in sinking fund	Net amount of loan
0					10,000.00
0.25	0	767.28	0	767.28	9,232.72
0.50	400.00	767.28	11.42	1,545.98	8,454.02
0.75	0	767.28	23.02	2,336.27	7,663.73
1.00	400.00	767.28	34.79	3,138.33	6,861.67
1.25	0	767.28	46.73	3,952.33	6,047.67
1.50	400.00	767.28	58.85	4,778.45	5,221.55
1.75	0	767.28	71.15	5,616.88	4,383.12
2.00	400.00	767.28	83.63	6,467.78	3,532.22
2.25	0	767.28	96.30	7,331.36	2,668.64
2.50	400.00	767.28	109.16	8,207.79	1,792.21
2.75	0	767.28	122.21	9,097.27	902.73
3.00	400.00	767.28	135.45	10,000.00	0.00

Practice Problems

Problem 39.1
A loan of 10,000 is being repaid with annual payments for 10 years using the sinking fund method. The loan charges 10% interest and the sinking fund earns 8%.
(a) Calculate the interest payment that is paid annually to service the loan.
(b) Calculate the sinking fund payment made annually.
(c) Calculate the amount in the sinking fund immediately after the deposit made at the end of 5 years.
(d) Create the sinking fund schedule for the loan.

Problem 39.2
If the loan in Problem 39.1 was repaid using the amortization method, but the annual payment was equal to the sum of the interest payment and the sinking fund deposit, calculate the interest rate under the amortization method.

Problem 39.3
A loan of 20,000 is being repaid with annual payments for 5 years using the sinking fund method. The loan charges 10% interest compounded twice a year. The sinking fund earns 8% compounded monthly. Calculate the interest payment that is paid annually to service the loan and the sinking fund deposit.

Problem 39.4
A loan of 20,000 is being repaid with monthly payments for 5 years using the sinking fund method. The loan charges 10% interest compounded twice a year. The sinking fund earns an annual effective interest rate of 8%.
(a) Calculate the interest payment that is paid monthly to service the loan and the sinking fund deposit paid monthly.
(b) Calculate the amount in the sinking fund immediately after the 30th payment.

Problem 39.5
Julie agrees to repay a loan of 10,000 using the sinking fund method over 10 years. The loan charges an annual effective interest rate of 7% while the sinking fund earns 6%.
Calculate the amount paid into the sinking fund each year less the amount of interest paid on the loan each year.

Problem 39.6
Kathy can take out a loan of 50,000 with Bank A or Bank B. With Bank A, she must repay the loan with 60 monthly payments using the amortization method with interest at 7% compounded

monthly. With Bank B, she can repay the loan with 60 monthly payments using the sinking fund method. The sinking fund will earn 6.5% compounded monthly.
What interest rate can Bank B charge on the loan so that Kathy's payment will be the same under either option?

Problem 39.7
Lauren is repaying a loan of 100,000 using the sinking fund method. At the end of each year she pays 7,000 into a sinking fund earning 8%. At the end of 5 years, Lauren pays off the loan using the sinking fund plus an additional payment of X. Calculate X.

Problem 39.8
Lauren is repaying a loan of 100,000 using the sinking fund method. At the end of each year she pays 7,000 into a sinking fund earning 8%. At the end of year Y, Lauren will have sufficient money in the sinking fund to repay the loan. Calculate Y.

Problem 39.9
Ryan takes out a loan of 100,000 and agrees to repay it over 10 years using the sinking fund method. Ryan agrees to pay interest to the lender at the end of each year. The interest rate is $.01(11-t)$ in year t. The sinking fund will earn 5% per year.
Determine the amount in the sinking fund after 10 years if the total payment made by Ryan at the end of each year is 12,000.

Problem 39.10
On a loan of 10,000, interest at 9% effective must be paid at the end of each year. The borrower also deposits X at the beginning of each year into a sinking fund earning 7% effective. At the end of 10 years the sinking fund is exactly sufficient to pay off the loan. Calculate X.

Problem 39.11
A borrower is repaying a loan with 10 annual payments of 1,000. Half of the loan is repaid by the amortization method at 5% effective. The other half of the loan is repaid by the sinking fund method, in which the lender receives 5% effective on the investment and the sinking fund accumulates at 4% effective. Find the amount of the loan.

Problem 39.12
A borrows 12,000 for 10 years, and agrees to make semiannual payments of 1,000. The lender receives 12% convertible semiannually on the investment each year for the first 5 years and 10% convertible semiannually for the second 5 years. The balance of each payment is invested in a sinking fund earning 8% convertible semiannually. Find the amount by which the sinking fund is short of repaying the loan at the end of the 10 years.

Problem 39.13
A borrower takes out a loan of 3000 for 10 years at 8% convertible semiannually. The borrower replaces one-third of the principal in a sinking fund earning 5% convertible semiannually, and the other two-thirds in a sinking fund earning 7% convertible semiannually. Find the total semiannual payment.

Problem 39.14
A payment of 36,000 is made at the end of each year for 31 years to repay a loan of 400,000. If the borrower replaces the capital by means of a sinking fund earning 3% effective, find the effective rate paid to the lender on the loan.

Problem 39.15
A loan of 30,000 is to be repaid using the sinking fund method over 6 years. The interest on the loan is paid at the end of each year and the interest rate is 10%. The sinking fund payment is made at the beginning of each year with the sinking fund earning 6%.
Calculate the amount paid into the sinking fund each year.

Problem 39.16
You have two equivalent ways to repay a loan of $100,000 over 20 years.
(a) Using the sinking fund method, where the rate of interest on the loan is 8% and the rate of interest earned by the sinking fund is 6%.
(b) Using the amortization method where the rate of interest is k.
Find k.

Problem 39.17 ‡
A 12-year loan of 8000 is to be repaid with payments to the lender of 800 at the end of each year and deposits of X at the end of each year into a sinking fund. Interest on the loan is charged at an 8% annual effective rate. The sinking fund annual effective interest rate is 4%. Calculate X.

Problem 39.18 ‡
A 20-year loan of 20,000 may be repaid under the following two methods:
i) amortization method with equal annual payments at an annual effective rate of 6.5%
ii) sinking fund method in which the lender receives an annual effective rate of 8% and the sinking fund earns an annual effective rate of j
Both methods require a payment of X to be made at the end of each year for 20 years. Calculate j.

Problem 39.19 ‡
Lori borrows 10,000 for 10 years at an annual effective interest rate of 9%. At the end of each year, she pays the interest on the loan and deposits the level amount necessary to repay the principal to a sinking fund earning an annual effective interest rate of 8%. The total payments made by Lori over the 10-year period is X. Calculate X.

Problem 39.20
John borrows 10,000 for 10 years and uses a sinking fund to repay the principal. The sinking fund deposits earn an annual effective interest rate of 5%. The total required payment for both the interest and the sinking fund deposit made at the end of each year is 1445.04. Calculate the annual effective interest rate charged on the loan.

Problem 39.21
Joe repays a loan of 10,000 by establishing a sinking fund and making 20 equal payments at the end of each year. The sinking fund earns 7% effective annually. Immediately after the fifth payment, the yield on the sinking fund increases to 8% effective annually. At that time, Joe adjusts his sinking fund payment to X so that the sinking fund will accumulate to 10,000 20 years after the original loan date. Determine X.

Problem 39.22
A corporation borrows 10,000 for 25 years, at an effective annual interest rate of 5%. A sinking fund is used to accumulate the principal by means of 25 annual deposits earning an effective annual interest rate of 4%. Calculate the sum of the net amount of interest paid in the 13^{th} installment and the increment in the sinking fund for the ninth year.

Problem 39.23
A loan of 1,000 is taken out at an annual effective interest rate of 5%. Level annual interest payments are made at the end of each year for 10 years, and the principal amount is repaid at the end of 10 years. At the end of each year, the borrower makes level annual payments to a sinking fund that earns interest at an annual effective rate of 4%. At the end of 10 years the sinking fund accumulates to the loan principal. Calculate the difference between the interest payment on the loan and the interest earned by the sinking fund in the fifth year.

Problem 39.24
John borrows 10,000 for 10 years at an annual effective interest rate of i. He accumulates the amount necessary to repay the loan by using a sinking fund. He makes 10 payments of X at the end of each year, which includes interest on the loan and the payment into the sinking fund, which earns an annual effective rate of 8%. If the annual effective rate of the loan had been $2i$, his total annual payment would have been $1.5X$. Calculate i.

Problem 39.25
Jason and Margaret each take out a 17 year loan of L. Jason repays his loan using the amortization method, at an annual effective interest rate of i. He makes an annual payment of 500 at the end of each year. Margaret repays her loan using the sinking fund method. She pays interest annually, also at an annual effective interest rate of i. In addition, Margaret makes level annual deposits at

the end of each year for 17 years into a sinking fund. The annual effective rate on the sinking fund is 4.62% and she pays off the loan after 17 years. Margaret's total payment each year is equal to 10% of the original loan amount. Calculate L.

Problem 39.26
A 10 year loan of 10,000 is to be repaid with payments at the end of each year consisting of interest on the loan and a sinking fund deposit. Interest on the loan is charged at a 12% annual effective rate. The sinking fund's annual effective interest rate is 8%. However, beginning in the sixth year the annual effective interest rate on the sinking fund drops to 6%. As a result, the annual payment to the sinking fund is then increased by X. Calculate X.

Problem 39.27
NTL Corp. has just taken out a 15 year, 200,000 loan on which it has to make semi-annual interest payments of 7,000.
As part of the loan agreement, NTL Corp. needs to make a deposit at the end of every 6 months into a sinking fund in order to retire the loan at the end of the 15th year. NTL Corp. plans to make semi-annual deposits of 4,929.98 into the sinking fund.
(a) What effective annual interest rate is NTL Corp. assuming it can earn on the sinking fund?
(b) What is the net interest cost in the 6 months following the 10th sinking fund deposit?

Problem 39.28
E Corp has a \$100,000 loan outstanding on which it has been paying semi-annual interest payments of \$4,000. E Corp has also been accumulating deposits made at the end of every 6 months in a sinking fund earning 5% effective annual interest so that it can retire the loan at the end of the 15th year. The lender has offered to accept 103% of the sinking fund balance immediately after the 29th deposit in exchange for the outstanding loan balance and the remaining loan interest payment. At what effective annual rate was the lender calculating the present value of the remaining amounts in order to make this offer equivalent in value?

Problem 39.29
A borrower takes out a loan of \$2,000 for two years. Create a sinking fund schedule if the lender receives 10% effective on the loan and if the borrower replaces the amount of the loan with semiannual deposits in a sinking fund earning 8% convertible quarterly.

Problem 39.30
John borrows \$10,000 for five years at 12% convertible semi-annually. John replaces the principal by means of deposits at the end of every year for five years into a sinking fund which earns 8% effective. Find the total dollar amount that John must pay over the five-year period to completely repay the loan.

Problem 39.31
A borrower is repaying a loan of 300,000 by the sinking fund method. The sinking fund earns an annual effective interest rate of 6.75%. Payments of \$22,520 are made at the end of each year for 20 years to repay the loan. These payments consist of both the interest payment to the lender and also the sinking fund deposit. What is the annual effective interest rate paid to the lender of the loan?

40 Loans Payable at a Different Frequency than Interest is Convertible

In Sections 37 - 39, we considered loans where the payment period coincides with the interest conversion period. In this section we examine loans where the payments are made at a different frequency than interest is convertible. The same observations made about the amortization method and the sinking fund method apply to the loans considered in this section. We will restrict our discussion to payments made at the end of an interest conversion period. A similar argument holds for payments made at the beginning of an interest conversion period.

Consider first the amortization schedule of a loan with payments made less frequently than interest is convertible. Consider a loan of $\frac{a_{\overline{n}|}}{s_{\overline{k}|}}$ at interest rate i per period being repaid with payments of 1 at the end of each k interest conversion periods for a total of n interest conversion periods. Thus, the total number of payments is $\frac{n}{k}$ which we assume is an integral number.

At the end of k interest conversion periods a payment of 1 is made, the interest paid is $[(1+i)^k - 1]\frac{a_{\overline{n}|}}{s_{\overline{k}|}} = 1-\nu^n$ so that the principal repaid is ν^n, and the outstanding loan balance is $\frac{a_{\overline{n}|}}{s_{\overline{k}|}} - \nu^n = \frac{a_{\overline{n-k}|}}{s_{\overline{k}|}}$. Next, at the end of $2k$ interest conversion periods, the interest paid is $[(1+i)^k-1]\frac{a_{\overline{n-k}|}}{s_{\overline{k}|}} = 1-\nu^{n-k}$ so that the principal repaid is ν^{n-k}, and the outstanding loan balance is $\frac{a_{\overline{n-k}|}}{s_{\overline{k}|}} - \nu^{n-k} = \frac{a_{\overline{n-2k}|}}{s_{\overline{k}|}}$. Continuing this process, we see that at the end of period mk, the interest paid is $[(1+i)^k - 1]\frac{a_{\overline{n-(m-1)k}|}}{s_{\overline{k}|}} = 1-\nu^{n-(m-1)k}$ and the principal repaid is $\nu^{n-(m-1)k}$. The outstanding loan balance is $\frac{a_{\overline{n-(m-1)k}|}}{s_{\overline{k}|}} - \nu^{n-(m-1)k} = \frac{a_{\overline{n-mk}|}}{s_{\overline{k}|}}$. The amortization table is shown below.

Period	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				$\frac{a_{\overline{n}\|}}{s_{\overline{k}\|}}$
k	1	$[(1+i)^k-1]\frac{a_{\overline{n}\|}}{s_{\overline{k}\|}} = 1-\nu^n$	ν^n	$\frac{a_{\overline{n}\|}}{s_{\overline{k}\|}} - \nu^n = \frac{a_{\overline{n-k}\|}}{s_{\overline{k}\|}}$
$2k$	1	$[(1+i)^k-1]\frac{a_{\overline{n-k}\|}}{s_{\overline{k}\|}} = 1-\nu^{n-k}$	ν^{n-k}	$\frac{a_{\overline{n-k}\|}}{s_{\overline{k}\|}} - \nu^{n-k} = \frac{a_{\overline{n-2k}\|}}{s_{\overline{k}\|}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
mk	1	$[(1+i)^k-1]\frac{a_{\overline{n-(m-1)k}\|}}{s_{\overline{k}\|}} = 1-\nu^{n-(m-1)k}$	$\nu^{n-(m-1)k}$	$\frac{a_{\overline{n-(m-1)k}\|}}{s_{\overline{k}\|}} - \nu^{n-(m-1)k} = \frac{a_{\overline{n-mk}\|}}{s_{\overline{k}\|}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-k$	1	$[(1+i)^k-1]\frac{a_{\overline{2k}\|}}{s_{\overline{k}\|}} = 1-\nu^{2k}$	ν^{2k}	$\frac{a_{\overline{2k}\|}}{s_{\overline{k}\|}} - \nu^{2k} = \frac{a_{\overline{k}\|}}{s_{\overline{k}\|}}$
n	1	$[(1+i)^k-1]\frac{a_{\overline{k}\|}}{s_{\overline{k}\|}} = 1-\nu^k$	ν^k	$\frac{a_{\overline{k}\|}}{s_{\overline{k}\|}} - \nu^k = 0$
Total	$\frac{n}{k}$	$\frac{n}{k} - \frac{a_{\overline{n}\|}}{s_{\overline{k}\|}}$	$\frac{a_{\overline{n}\|}}{s_{\overline{k}\|}}$	

Note that the principal repaid column is a geometric progression with common ratio $(1+i)^k$.

Example 40.1
A loan of \$15,000 is to be repaid by means of six annual payments of \$3719.18 each. The nominal interest rate is 12% compounded monthly. Create an amortization schedule for this transaction.

Solution.
The schedule is given below.

Period	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				15,000
12	3719.18	1902.37	1816.81	13,183.19
24	3719.18	1671.96	2047.22	11,135.97
36	3719.18	1412.32	2306.86	8829.11
48	3719.18	1119.75	2599.43	6229.68
60	3719.18	790.08	2929.10	3300.58
72	3719.18	418.60	3300.58	0

Next, consider a loan of $a_{\overline{n}|}^{(m)}$ at interest rate i per period being repaid with payments of $\frac{1}{m}$ at the end of each mth of an interest conversion period for a total of n interest conversion periods. Thus, the total number of payments is mn which we assume is an integral number.
At the end of the first mth of an interest conversion period a payment of $\frac{1}{m}$ is made, the interest paid is $\frac{i}{i^{(m)}}a_{\overline{n}|}^{(m)} = \frac{1}{m}(1-\nu^n)$ so that the principal repaid is $\frac{1}{m}\nu^n$, and the outstanding loan balance is $a_{\overline{n}|}^{(m)} - \frac{1}{m}\nu^n = a_{\overline{n-\frac{1}{m}}|}^{(m)}$. Next, at the end of second mth of an interest conversion period, the interest paid is $\frac{i}{i^{(m)}}a_{\overline{n-\frac{1}{m}}|}^{(m)} = \frac{1}{m}(1-\nu^{n-\frac{1}{m}})$ so that the principal repaid is $\frac{1}{m}\nu^{n-\frac{1}{m}}$, and the outstanding loan balance is $a_{\overline{n-\frac{1}{m}}|}^{(m)} - \frac{1}{m}\nu^{n-\frac{1}{m}} = a_{\overline{n-\frac{2}{m}}|}^{(m)}$. Continuing this process, we see that at the end of period $\frac{t}{n}$, the interest paid is $\frac{i}{i^{(m)}}a_{\overline{n-\frac{(t-1)}{m}}|}^{(m)} = \frac{1}{m}(1-\nu^{n-\frac{t-1}{m}})$ and the principal repaid is $\frac{1}{m}\nu^{n-\frac{(t-1)}{m}}$. The outstanding loan balance is $a_{\overline{n-\frac{t-1}{m}}|}^{(m)} - \frac{1}{m}\nu^{n-\frac{t-1}{m}} = a_{\overline{n-\frac{t}{m}}|}^{(m)}$. The amortization table is shown below.

Period	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				$a_{\overline{n}\rceil}^{(m)}$
$\frac{1}{m}$	$\frac{1}{m}$	$\frac{i}{i^{(m)}}a_{\overline{n}\rceil}^{(m)} = \frac{1}{m}(1-\nu^n)$	$\frac{1}{m}\nu^n$	$a_{\overline{n}\rceil}^{(m)} - \frac{1}{m}\nu^n = a_{\overline{n-\frac{1}{m}}\rceil}^{(m)}$
$\frac{2}{m}$	$\frac{1}{m}$	$\frac{i}{i^{(m)}}a_{\overline{n-\frac{1}{m}}\rceil}^{(m)} = \frac{1}{m}(1-\nu^{n-\frac{1}{m}})$	$\frac{1}{m}\nu^{n-\frac{1}{m}}$	$a_{\overline{n-\frac{1}{m}}\rceil}^{(m)} - \frac{1}{m}\nu^{n-\frac{1}{m}} = a_{\overline{n-\frac{2}{m}}\rceil}^{(m)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{t}{m}$	$\frac{1}{m}$	$\frac{i}{i^{(m)}}a_{\overline{n-\frac{(t-1)}{m}}\rceil}^{(m)} = \frac{1}{m}(1-\nu^{n-\frac{t-1}{m}})$	$\frac{1}{m}\nu^{n-\frac{(t-1)}{m}}$	$a_{\overline{n-\frac{t-1}{m}}\rceil}^{(m)} - \frac{1}{m}\nu^{n-\frac{t-1}{m}} = a_{\overline{n-\frac{t}{m}}\rceil}^{(m)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-\frac{1}{m}$	$\frac{1}{m}$	$\frac{i}{i^{(m)}}a_{\overline{\frac{2}{m}}\rceil}^{(m)} = \frac{1}{m}(1-\nu^{\frac{2}{m}})$	$\frac{1}{m}\nu^{\frac{2}{m}}$	$a_{\overline{n-\frac{2}{m}}\rceil}^{(m)} - \frac{1}{m}\nu^{n-\frac{2}{m}} = a_{\overline{n-\frac{1}{m}}\rceil}^{(m)}$
n	$\frac{1}{m}$	$\frac{i}{i^{(m)}}a_{\overline{\frac{1}{m}}\rceil}^{(m)} = \frac{1}{m}(1-\nu^{\frac{1}{m}})$	$\frac{1}{m}\nu^{\frac{1}{m}}$	$a_{\overline{n-\frac{1}{m}}\rceil}^{(m)} - \frac{1}{m}\nu^{n-\frac{1}{m}} = a_{\overline{n-\frac{1}{m}}\rceil}^{(m)} - \frac{1}{m}\nu^{\frac{1}{m}} = 0$
Total	n	$n - a_{\overline{n}\rceil}^{(m)}$	$a_{\overline{n}\rceil}^{(m)}$	

Note that the principal repaid column is a geometric progression with common ratio $(1+i)^{\frac{1}{m}}$.

Example 40.2
A debt is being amortized by means of monthly payments at an annual effective rate of interest of 11%. If the amount of principal in the third payment is \$1000, find the amount of principal in the 33rd payment.

Solution.
Recall that the principal repaid column is a geometric progression with common ratio $(1+i)^{\frac{1}{m}}$. The interval of time from the 3rd payment to the 33rd payment is $\frac{33-3}{12} = 2.5$ years. Thus, the principal in the 33rd payment is

$$1000(1.11)^{2.5} = \$1298.10 \blacksquare$$

Example 40.3
A loan of \$10,000 is to be repaid by means of twelve monthly payments. The nominal interest rate is 12% compounded quarterly. Create an amortization schedule for this transaction.

Solution.
Let R be the amount of monthly payment. Then R satisfies the equation $10000 = 3Ra_{\overline{4}\rceil 0.03}^{(3)}$. Solving this equation for R we find $R = \$887.94$. Moreover, we have

$$\frac{i^{(3)}}{3}a_{\overline{4}\rceil 0.03}^{(3)} = \frac{0.02970490211}{3} = 99.02.$$

The schedule is given below.

Period	Payment amount	Interest paid	Principal repaid	Outstanding loan balance
0				10,000
$\frac{1}{3}$	887.94	99.02	788.92	9211.08
$\frac{2}{3}$	887.94	91.20	796.74	8414.34
1	887.94	83.32	804.62	7609.72
$\frac{4}{3}$	887.94	75.35	812.59	6797.13
$\frac{5}{3}$	887.94	67.30	820.64	5976.49
2	887.94	59.18	828.76	5147.73
$\frac{7}{3}$	887.94	50.97	836.97	4310.76
$\frac{8}{3}$	887.94	42.68	845.26	3465.50
3	887.94	34.31	853.63	2611.87
$\frac{10}{3}$	887.94	25.86	862.08	1749.79
$\frac{11}{3}$	887.94	17.33	870.61	879.18
4	887.94	8.71	879.23	0

It is not recommended that the reader depends upon the memorization of formulas in amortization and sinking fund schedules. It is rather more preferable to create these schedules using the basic principles. We illustrate the use of basic principles in creating a sinking fund schedule in the next example.

Example 40.4
John borrows \$2000 for two years at an annual effective interest rate of 10%. He replaces the principal by means of semiannual deposits for two years in a sinking fund that earns 8% convertible quarterly. Create a sinking fund schedule for this transaction.

Solution.
The interest payment on the loan is $2000 \times 10\% = \$200$ at the end of each year. If R is the semiannual deposit in the sinking fund then

$$R\frac{s_{\overline{8}|0.02}}{s_{\overline{2}|0.02}} = 2000.$$

Solving this equation for R we find

$$R = 2000\frac{s_{\overline{2}|0.02}}{s_{\overline{8}|0.02}} = \$470.70.$$

The sinking fund schedule is given below.

Period	Interest paid	Sinking fund deposit	Interest earned on sinking fund	Amount in sinking fund	Net amount of loan
0					
1/4	0	0	0	0	2000.00
1/2	0	470.70	0	470.70	1529.30
3/4	0	0	9.41	480.11	1519.89
1	200.00	470.70	9.60	960.41	1039.59
1 1/4	0	0	19.21	979.62	1020.38
1 1/2	0	470.70	19.59	1469.91	530.09
1 3/4	0	0	29.40	1499.31	500.69
2	200.00	470.70	29.99	2000.00	0

Practice Problems

Problem 40.1
A debt is being amortized by means of annual payments at a nominal interest rate of 12% compounded semiannually. If the amount of principal in the third payment is \$1000, find the amount of principal in the 33rd payment.

Problem 40.2
A borrower is repaying a loan by making quarterly payments of \$500 over 10 years. Portion of each payment is principal and the rest is interest. Assume an annual effective rate of 8%, how much interest is paid by the borrower over the 10-year period?

Problem 40.3
A lender receives payments of \$3000 at the end of every year over an unknown number of years. If the interest portion of the third payment is \$2000, find the amount of principal in the sixth payment. Assume a nominal interest rate of 10% payable quarterly.

Problem 40.4
A borrows \$10,000 for five years at 12% convertible semiannually. A replaces the principal by means of deposits at the end of every year for five years into a sinking fund which earns 8% effective. Find the total dollar amount which A must pay over the five-year period to completely repay the loan.

Problem 40.5
A borrows \$5000 for 10 years at 10% convertible quarterly. A does not pay interest currently and will pay all accrued interest at the end of 10 years together with the principal. Find the annual sinking fund deposit necessary to liquidate the loan at the end of 10 years if the sinking fund earns 7% convertible quarterly.

Problem 40.6
A borrows \$10,000 for five years at 12% convertible semiannually. A replaces the principal by means of deposits at the end of every year for five years into a sinking fund which earns 8% effective. Create a sinking fund schedule for this loan.

41 Amortization with Varying Series of Payments

In this section, we consider amortization methods with more general patterns of payments, not necessarily leveled. We will keep assuming that the interest conversion period and the payment period are equal and coincide.
Consider a loan L to be repaid with n periodic payments (that include principal and interest) $R_1, R_2, \cdots, R_n$. The equation of value at time $t = 0$ is

$$L = \sum_{t=1}^{n} \nu^t R_t.$$

In most cases the series of payments R_t follows some regular pattern encountered with annuties so that the results of Section 26 can be used.

Example 41.1
A borrower is repaying a loan with payments at the end of each year for 10 years, such that the payment the first year is \$200, the second year is \$190, and so forth, until the 10th year it is \$110. Find an expression of the amount of the loan.

Solution.
The amount of the loan is

$$L = 100a_{\overline{10}|} + 10(Da)_{\overline{10}|} \blacksquare$$

For the type of installments considered in this section, amortization schedules can be constructed from first principles as discussed in Section 38. Furthermore, the outstanding loan balance column can be found retrospectively or prospectively as in Section 37, from which the remaining columns of interest paid and principal repaid can be found.

Example 41.2
A loan is repaid with payments which start at \$200 the first year and increase by \$50 per year until a payment of \$1000 is made, at which time payments cease. If the interest is 4% effective, find the amount of interest and principal paid in the fourth payment.

Solution.
The fourth payment is $R_4 = \$350.00$. The outstanding loan balance after making the third payment is

$$B_3^p = 300a_{\overline{14}|} + 50(Ia)_{\overline{14}|}.$$

The interest paid after the fourth payment is

$$\begin{aligned}I_4 =& iB_3^p = 300(1-\nu^{14}) + 50(\ddot{a}_{\overline{14}|} - 14\nu^{14})\\ =&300(1-0.57747)+50(10.98565-8.08465)\\ =&\$271.81\end{aligned}$$

The principal portion of the fourth payment is

$$P_4 = R_4 - I_4 = 350.00 - 271.81 = \$78.19 \blacksquare$$

One common pattern of periodic installments is when the borrower makes level payments of principal. Clearly, this leads to successive total payments (consisting of interest and principal) to decrease due to the decrease in successive outstanding loan balance (which in turn results in a decrease in successive interest paid). We illustrate this in the following example.

Example 41.3
A borrower is repaying a \$1000 loan with 10 equal payments of principal. Interest at 6% compounded semiannually is paid on the outstanding balance each year. Find the price to yield an investor 10% convertible semiannually.

Solution.
The period principal payment is \$100 each. The periodic interest payments are $:30, 27, 24, \cdots, 9, 3$. Thus, the price at time $t = 0$ is just the present value of the principal and interest paid. That is,

$$\text{Price} = 100a_{\overline{10}|0.05} + 3(Da)_{\overline{10}|0.05} = 908.87 \blacksquare$$

It is possible that when using varying payments, the loan payment for a period is less than the interest charged over that period. This leads to an increase in the outstanding balance which in turn means a negative principal. In finance, such a case is referred to as **negative amortization** or **deferred interest**. Credit card interest rates and payment plans are examples of negative amortization. Most credit cards carry high interest rates yet require a low minimum payment.

Example 41.4
A loan of 1000 is being repaid with annual payments over 10 years. The payments in the last five years are 5 times the payments in the first 5 years. If $i = 0.08$, calculate the principal amortized in the fifth payment.

Solution.
Let K be the size of the payment in the first 5 years. Then

$$1000 = K\frac{(1-1.08^{-5})}{0.08} + 5K(1.08^{-5})\frac{(1-1.08^{-5})}{0.08}$$

or $1000 = 17.57956685K \to K = 56.88422296$.
The outstanding loan balance after making the 4th payment is

$$\frac{56.88422296}{1.08} + 5(56.88422296)(1.08^{-1})\frac{(1-1.08^{-5})}{0.08} = 1104.16228.$$

The outstanding loan balance after making the 5th payment is

$$5(56.88422296)\frac{(1-1.08^{-5})}{0.08} = 1135.61104.$$

The principal amortized in the fifth payment is $1104.16228 - 1135.61104 = -\$31.45$ ■

Next, consider a varying series of payments with the sinking fund method. We will assume that the interest paid to the lender is constant each period so that only the sinking fund deposits vary.
Let the varying payments be denoted by $R_1, R_2, \cdots, R_n$ and assume that $i \neq j$. Let L denote the amount of the loan. Then the sinking fund deposit for the tth period is $R_t - iL$. Since the accumulated value of the sinking fund at the end of n periods must be L, we have

$$\begin{aligned} L =&(R_1 - iL)(1+j)^{n-1} + (R_2 - iL)(1+j)^{n-2} + \cdots + (R_n - iL) \\ =&\sum_{t=1}^{n} R_t(1+j)^{n-t} - iLs_{\overline{n}|j} \end{aligned}$$

where i is rate of interest paid on the loan and j rate of interest earned on the sinking fund.
Solving for L we obtain

$$L = \frac{\sum_{t=1}^{n} R_t(1+j)^{n-t}}{1 + is_{\overline{n}|j}} = \frac{\sum_{t=1}^{n} \nu_j^t R_t}{1 + (i-j)a_{\overline{n}|j}}.$$

Note that if $i = j$ then $L = \sum_{t=1}^{n} R_t \nu^t$. Thus, the amortization method and the sinking fund method are equivalent.

Example 41.5
A borrower is repaying a loan at 5% effective with payments at the end of each year for 10 years, such that the payment the first year is \$200, the second year is \$190, and so forth, until the 10th year it is \$110. Find the amount of the loan if the borrower pays 6% effective on the loan and accumulates a sinking fund to replace the amount fo the loan at 5% effective.

Solution.
We have

$$L = \frac{100a_{\overline{10}|0.05} + 10(Da)_{\overline{10}|0.05}}{1 + (0.06 - 0.05)a_{\overline{10}|0.05}} = \frac{1227.83}{1 + (0.01)(7.7217)} = \$1139.82$$ ■

Practice Problems

Problem 41.1
A borrows \$10,000 from B and agrees to repay it with a series of 10 installments at the end of each year such that each installment is 20% greater than the preceding installment. The rate of interest on the loan is 10% effective. Find the amount of principal repaid in the first three installments.

Problem 41.2
John borrows 10,000 from Tony and agrees to repay the loan with 10 equal annual installments of principal plus interest on the unpaid balance at 5% effective. Immediately after the loan is made, Tony sells the loan to Chris for a price which is equal to the present value of John's future payments calculated at 4%. Calculate the price that Chris will pay for the loan.

Problem 41.3
A loan is being repaid with 10 payments of $1000t$ at the end of year t. (In other words, the first payment is 1000, the second payment is 2000, etc.) The interest rate on the loan is 6%.
Calculate the outstanding principal immediately after the third payment.

Problem 41.4
A loan of 100,000 is being repaid with annual payments at the end of each year for 10 years. The interest rate on the loan is 10.25%. Each annual payment increases by 5% over the previous annual payment.
Calculate the principal in the fifth payment.

Problem 41.5 ‡
A 30-year loan of 1000 is repaid with payments at the end of each year. Each of the first ten payments equals the amount of interest due. Each of the next ten payments equals 150% of the amount of interest due. Each of the last ten payments is X. The lender charges interest at an annual effective rate of 10%. Calculate X.

Problem 41.6 ‡
A loan is repaid with 20 increasing annual installments of $1,2,3,\cdots,20$. The payments begin one year after the loan is made. Find the principal contained in the 10th payment, if the annual interest rate is 4%.

Problem 41.7
In order to repay a school loan, a payment schedule of 200 at the end of the year for the first 5 years, 1200 at the end of the year for the next 5 years, and 2200 at the end of the year for the final 5 years is agreed upon. If interest is at the annual effective rate of $i = 6\%$, what is the loan value?

Problem 41.8
Create an amortization schedule of the previous problem for the first seven years.

Problem 41.9
A borrows \$20,000 from B and agrees to repay it with 20 equal annual installments of principal in addition to the interest on the unpaid balance at 3% effective. After 10 years B sells the right to future payments to C, at a price which yields C 5% effective over the next 5 years and 4% effective over the final 5 years. Find the price which C should pay to the nearest dollar.

Problem 41.10
A loan of 3000 at an effective quarterly interest rate of $j = .02$ is amortized by means of 12 quarterly payments , beginning one quarter after the loan is made. Each payment consists of a principal repayment of 250 plus interest due on the previous quarter's outstanding balance. Construct the amortization schedule.

Problem 41.11
A loan is being repaid with 10 annual payments. The first payment is equal to the interest due only, the second payment is twice the first, the third payment is three time the first, and so forth. Prove that at the rate of interest on the loan is: $(Ia)_{\overline{10}|} = a_{\overline{\infty}|}$.

Problem 41.12
A loan is being repaid with 10 payments. The first payment is 10, the second 9, the third is 8, and so forth with the tenth payment being 1. Find an expression for the amount of interest in the sixth payment.

Problem 41.13
A has money invested at effective rate i. At the end of the first year A withdraws 162.5% of the interest earned; at the end of the second year A withdraws 325% of the interest earned, and so forth with the withdrawal factor increasing in arithmetic progression. At the end of 16 years the fund is exhausted. Find i.

Problem 41.14
A loan is amortized over five years with monthly payments at a nominal interest rate of 9% compounded monthly. The first payment is 1000 and is to be paid one month from the date of the loan. Each succeeding monthly payment will be 2% lower than the prior payment. Calculate the outstanding loan balance immediately after the 40th payment is made.

Problem 41.15
June borrows 20,000 from April and agrees to repay it with a series of ten installments at the end

of each year, such that each installment is 15% greater than the preceding installment. The rate of interest on the loan is 10% annual effective. Let P_j denote the amount of principal repaid in year j. Calculate $P_1 + P_2$.

Problem 41.16
Smith borrows 30,000 at an annual effective interest rate of 10%. He agrees to make annual payments at the end of each year for 9 years, and an additional balloon payment, X, at the end of the tenth year. Each of the first 9 payments equals 25% more than what is owed in interest at the time of the payment. The balloon payment, X, equals the amount needed to pay off the loan. Determine X.

Problem 41.17
A loan is being repaid by the amortization method using 10 semiannual payments with the first payment due six months after the loan is made. The first payment is 50 and each subsequent payment is 50 more than its previous payment. The interest rate on the loan is 10%, compounded semiannually. Determine the amount of principle repaid in the seventh payment.

Problem 41.18
A 10 year loan with an effective annual interest rate of 5% is to be repaid with the following payments
(1) 100 at the end of the second year
(2) 200 at the end of the fourth year
(3) 300 at the end of the sixth year
(4) 400 at the end of the eighth year
(5) 500 at the end of the tenth year
Calculate the amount of interest included in the second payment.

Problem 41.19
John takes out a 10 year loan. The loan is repaid by making 10 annual repayments at the end of each year. The first loan repayment is equal to X, with each subsequent repayment 10.16% greater than the previous year's repayment. The annual effective interest rate being charged on the loan is 8%. The amount of interest repaid during the first year is equal to 892.20. Calculate X.

Problem 41.20
Don takes out a 10 year loan of L which he repays with annual payments at the end of each year using the amortization method. Interest on the loan is charged at an annual effective rate of i. Don repays the loan with a decreasing series of payments. He repays 1000 in year one, 900 in year two, 800 in year three, $\cdots$, and 100 in year ten. The amount of principal repaid in year three is equal to 600. Calculate L.

Problem 41.21 ‡
Betty borrows 19,800 from Bank X. Betty repays the loan by making 36 equal payments of principal at the end of each month. She also pays interest on the unpaid balance each month at a nominal rate of 12%, compounded monthly.
Immediately after the 16th payment is made, Bank X sells the rights to future payments to Bank Y. Bank Y wishes to yield a nominal rate of 14%, compounded semi-annually, on its investment. What price does Bank X receive?

Problem 41.22
Two loans for equal amounts are repaid at an effective interest rate of 4%. Loan A is repaid with 30 equal annual payments. Loan B is to be repaid by 30 annual payments, each containing equal principal amounts and an interest amount based on the unpaid balance.
Payments are made at the end of each year. The annual payment for Loan A first exceeds the annual payment for Loan B with the kth payment. Find k.

Problem 41.23
A borrows \$2000 at an effective rate of interest of 10% per annum and agrees to repay the loan with payments at the end of each year. The first payment is to be \$400 and each payment thereafter is to be 4% greater than the preceding payment, with a smaller final payment made one year after the last regular payment.
(a) Find the outstanding loan balance at the end of three years.
(b) Find the principal repaid in the third payment.

Bonds and Related Topics

This chapter is about bonds and the valuation of bonds. We will learn how to determine the price of a bond that should be paid by an investor for a desired given yield rate, determine the yield rate given the price, how a bond is amortized, and finally determine the value of a bond on a given date after it has been purchased.

42 Types of Bonds

Perhaps the simplest way for a company or a government agency to raise cash is for them to sell bonds to the public. A **bond** is an interest bearing security which promises to pay a stated amount (or amounts) of money at some future date (or dates). The company or government branch which is issuing the bond outlines how much money it would like to borrow and specifies a length of time, along with an interest rate it is willing to pay. Investors who then lend the requested money to the issuer become the issuer's creditors through the bonds that they hold.

The **term** of the bond is the length of time from the date of issue until the date of final payment. The date of the final payment is called the **maturity date**. Bonds with an infinite term are called **perpetuals.**

The detailed conditions of the bond may permit the bond to be called at some date prior to the maturity date, at which time the issuer (i.e., the lender) will repay the commitment, possibly with some additional amounts. Such a bond is **callable.**

Any date prior to, or including, the maturity date on which a bond may be redeemed is termed a **redemption date**.

The **par value** or **face value** of a bond is the amount that the issuer agrees to repay the bondholder by the maturity date.

There are several classification of bonds:

• **Accumulation bond** is one in which the redemption price includes the original loan plus accumulated interest. Examples of such bonds are the Series E Savings bonds.

• **Bonds with coupons** are periodic payments of interest made by the issuer of the bond prior to redemption. It is called a "coupon" because some bonds literally have coupons attached to them. Holders receive interest by stripping off the coupons and redeeming them. This is less common today as more records are kept electronically. In what follows, we we use the term "bond" we mean bonds with coupons. **Zero coupon bonds** are bonds that pay no periodic interest payments. It just pays a lump sum at redemption date.

• **Registered and Unregistered bonds.** A **registered bond** is a bond issued with the name of the owner printed on the face of the certificate. If the owner decides to sell the bond, the change must be reported to the borrower. The coupon payments are paid by the borrower to the owners of record on each coupon payment date. An **unregistered bond** or **bearer bond** is one in which the lender is not listed in the records of the borrower. In this case, the bond belongs to whomever has legal possession of it. Again, these bonds are occasionally called coupon bonds, due to the physically attached coupons.

• **Fixed-rate and floated-rate bonds.** A **fixed-rate bond** is a bond that has a fixed rate over the term of the bond. On the other hand, a bond that has a fluctuated interest rate over the term of the bond is called a **floating-rate bond.**

• **Mortgage and debenture bonds.** A **mortgage bond** is a more secured bond backed by a collateral such as a mortgage on a property. Mortgage bonds are backed by real estate or physical equipment that can be liquidated. A **debenture bond** is an unsecured bond issued by a civil or governmental corporation or agency and backed only by the credit standing of the issuer.

• **Income or adjustment bonds.** They are a type of high risk bonds in which coupons are paid only if the borrower has earned sufficient income to pay them.

• **Junk bonds.** A **junk bond** is a high-risk bond of default in payments. The risk that a bond issuer does not pay the coupon or principal payments is called **default risk.** Because of this risk, these bonds typically pay higher yields than better quality bonds in order to make them attractive to investors.

• **Convertible bond.** A **convertible bond** is a bond that can be converted into the common stock of the company at the option of the bond owner. The owner of the bond is compensated with the ability to convert the bond to common stock, usually at a substantial discount to the stock's market value.

• **Serial bonds.** A set of bonds issued at the same time but having different maturity dates. These are used when a borrower is in need of a large amount of money.

• **Treasury bonds.** Issued by the US Treasury. Terms of seven or more years.

• **Treasury bills.** Short term debt with maturities of 13, 26, or 52 weeks. T-bills yields are computed as rates of discount. These yields are computed on a simple discount basis. The basis for counting time periods for T-bills is actual/360.

• **Municipal bonds.** These are bonds issued by state and local governments to finance large, long−term capital projects (e.g., hospitals, highways, schools).

Example 42.1

A 10 year bond matures for its par value of 5000. The price of the bond is 4320.48 at an 8% yield convertible semi-annually. Calculate the coupon rate convertible semi-annually. Coupon payment is defined to be the product of the face value and the coupon periodic rate. Also, the price of a bond is the present value of all future payments.

Solution.

Let k be the semiannual effective coupon rate. Then

$$4320.48 = \frac{5000}{1.04^{20}} + 5000k\frac{(1-1.04^{-20})}{0.04}$$

or

$$5000k\frac{(1-1.04^{-20})}{0.04} = 2038.545269$$

Thus,

$$k = 2038.545269\frac{0.04}{[5000(1-1.04^{-20})]} = 0.029999$$

the coupon rate convertible semi-annually is $2k = 0.059999 = 6.00\%$ ■

Example 42.2
A $1000 par value bond bearing a 6% coupon rate payable semi-annually will be redeemed at 105% at the end of 15 years. Find the price to yield an investor 5% effective.

Solution.
The price is

$$P = 30a_{\overline{30}|j} + 1050(1.05)^{-15} = \$1135.54$$

where $j = (1.05)^{\frac{1}{2}} - 1$ ■

Example 42.3
A 10-year accumulation bond with an initial par value(i.e. face value) of 1000 earns interest of 8% compounded semiannually. Find the price to yield an investor 10% effective.

Solution.
The only return payment is at maturity. The price to yield 10% interest will therefore be

$$(1.10)^{-10}[1000(1.04)^{20}] = \$844.77$$ ■

Example 42.4
Find the price which should be paid for a zero coupon bond which matures for 1000 in 10 years to yield:
(a) 10% effective
(b) 9% effective
(c) Thus, a 10% reduction in the yield rate causes the price to increase by what percentage?

Solution.
(a) The bond is now worth $1000(1.10)^{-10} = \$385.54$
(b) The bond is now worth $1000(1.09)^{-10} = \$422.41$
(c) The 10% decrease in the interest rate thereby increases the price by

$$\frac{422.41 - 385.54}{385.54} = 9.6\%$$ ■

Practice Problems

Problem 42.1
A zero bond will pay \$1000 at the end of 10 years, and is currently selling for \$500. What is the annual return earned by the purchaser?

Problem 42.2
Steve purchases a 10-year zero-coupon bond for \$400 and has par value of \$1,000. Find the yield rate convertible semiannually that would earned by Steve.

Problem 42.3
John buys a 13-week T-bill with par value \$10,000 at a simple discount rate of 7.5%. What price did he pay for the bond?

Problem 42.4
An investor purchases a 26-week T-bill with face value of \$10,000 and with a discount yield of d for \$9,600.
(a) Find d.
(b) Even though T-bills are not usually quoted on an annual effective yield basis, the annual effective yield can still be determined. Find the annual effective rate of interest of this investment, assuming the investment period is exactly half a year.

Problem 42.5
An investor purchases a 26-week T-bill with face value \$10,000 for $\$X$ and earns an annual effective rate of 4.17%. Determine X.

Problem 42.6
An investor buys a zero-coupon bond with par value \$1,000 for \$493.63. The maturity date is Y years and the yield rate convertible semi-annually is 8%. Calculate Y.

Problem 42.7
A 20 year bond matures for its par value of 10,000. The coupon payable semi-annually is 400. Calculate the price of the bond at a 6% yield rate convertible semi-annually.

Problem 42.8
A 20 year bond matures for its par value of 10,000. The coupon payable semi-annually is 400. Calculate the price of the bond at a 6% annual effective yield rate.

Problem 42.9
A \$1000 par value bond bearing a 5% coupon rate payable semi-annually will be redeemed at 108% at the end of 7 years. Find the price to yield an investor 5% convertible semi-annually.

43 The Various Pricing Formulas of a Bond

In this section we consider the question of determining the purchase price which will produce a given yield rate to an investor. We have already encountered this question in the previous section. When considering the price of a bond, we make the following assumptions:
(1) All obligations will be paid by the bond issuer on the specified dates of payments. Any possibility of default will be ignored.
(2) The bond has a fixed maturity date.
(3) The price of the bond is decided immediately after a coupon payment date, or alternatively, at issue date if the bond is brand new.
The following symbols and notation will be used in connection with bonds valuation:

$P =$	the **price** paid for a bond.
$F =$	The **par value** or **face value** or **face amount.** This amount is is usually printed on the front of the bond and is often the amount payable at the maturity date.
$C =$	the **redemption value** of a bond, i.e. the amount of money paid at a redemption date to the holder of the bond.
$r =$	The **coupon rate** is the effective rate per coupon payment period used in determining the amount of the coupon. The default payment period is a half-year. For example, $r = 0.035$ for a 7% nominal coupon paid semi-annually.
$Fr =$	The **amount** of a coupon payment.
$g =$	The **modified coupon rate.** It is defined by $g = \frac{Fr}{C}$. Thus, g is the coupon rate per unit of redemption value, rather than per unit of par value.
$i =$	The **yield rate** of a bond, or the **yield to maturity.** The interest rate actually earned by the investor, assuming the bond is held until redemption or maturity.
$n =$	The number of coupon payment periods from the date of calculation until maturity date or redemption date
$K =$	the present value, computed at the yield rate, of the redemption value at the maturity date, or a redemption date, i.e. $K = C\nu^n$ at the yield rate i.
$G =$	The **base amount** of a bond. It is defined by $Gi = Fr$ or $G = \frac{Fr}{i}$. Thus, G is the amount which, if invested at the yield rate i, would produce periodic interest payments equal to the coupons on the bond.

The quantities F, C, r, g, and n are given by the terms of a bond and remain fixed throughout the bonds life. On the other hand, P and i will vary throughout the life of the bond. Price and yield rate have a precise inverse relationship to each other,i.e. as the price rises the yield rate falls and

vice-versa.
There are three different quoted yields associated with a bond:
(1) **Nominal yield** is the ratio of annualized coupon payment to par value. For example, 2 coupons per year of \$3.50 on a \$100 par value bond result in a 7.00% nominal yield. The reader should note that the use of the word "nominal" in this context is different than the one used in Section 9.
(2) **Current yield** is the ratio of the annualized coupon payment to the original price of the bond. For example, if you paid \$90 per \$100 of par value of the bond described above, the current yield would be $\frac{\$7.00}{\$90} = 7.78\%$
(3) **Yield to maturity** is the actual annualized yield rate, or internal rate of return.

Four formulas for price
Like loans, the price of a bond is defined to be the present value of all future payments. We shall consider four different ways of determining the present value, or price of a bond. The formulas are derivable one from the other: there are situations where one may be more useful than another for specific applications. The first of these is called the **basic formula**. It is found by writing the equation of value at time $t = 0$ of the time diagram given in Figure 43.1 and is given by

$$P = Fra_{\overline{n}|i} + C\nu_i^n = Fra_{\overline{n}|i} + K.$$

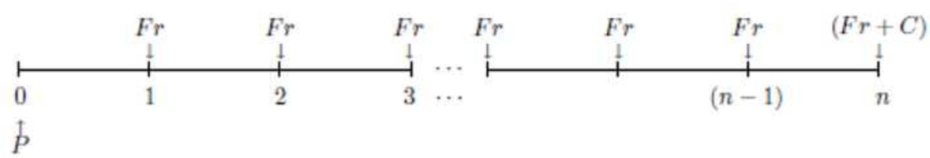


Figure 43.1

This formula has a verbal interpretation: the price of a bond is equal to the present value of future coupons plus the present value of the redemption value.
The second formula is the **premium/discount** formula, and is derived from the basic formula:

$$\begin{aligned} P =& Fra_{\overline{n}|i} + C\nu_i^n \\ =& Fra_{\overline{n}|i} + C(1 - ia_{\overline{n}|i}) \\ =& C + (Fr - Ci)a_{\overline{n}|i} \end{aligned}$$

This says that the price of a bond is the sum of the redemption value and the present value of amortization of premium or discount paid at purchase. Amortization of premium and discount will be discussed in the next section.

The third, the **base amount formula**, can also be derived from the first formula as follows:

$$\begin{aligned}P =& Fra_{\overline{n}|i} + C\nu_i^n \\ =& Gia_{\overline{n}|i} + C\nu_i^n \\ =& G(1-\nu_i^n) + C\nu_i^n \\ =& G + (C-G)\nu_i^n\end{aligned}$$

This says that the price is the sum of the base amount and the present value of the difference between base amount and redemption value.
The fourth, the **Makeham formula**, is also obtained from the basic formula:

$$\begin{aligned}P =& C\nu_i^n + Fra_{\overline{n}|i} \\ =& C\nu_i^n + Cg\left(\frac{1-\nu_i^n}{i}\right) \\ =& C\nu_i^n + \frac{g}{i}(C - C\nu_i^n) \\ =& K + \frac{g}{i}(C-K)\end{aligned}$$

Example 43.1
A 10-year 100 par value bond bearing a 10% coupon rate payable semi-annually, and redeemable at 105, is bought to yield 8% convertible semiannually. Find the price. Verify that all four formulas produce the same answer.

Solution.
We are given the following:

$$\begin{array}{ll} F = & 100 \\ C = & 105 \\ r = & 5\% \\ Fr = & 5 \\ g = \frac{Fr}{C} = & \frac{1}{21} \\ i = & 4\% \\ n = & 20 \\ K = C\nu^n = & 105(1.04)^{-20} = 47.92 \\ G = \frac{Fr}{i} = & 125 \end{array}$$

Using the basic formula we find

$$P = Fra_{\overline{n}|} + K = 5a_{\overline{20}|0.04} + 47.92 = 5(13.59031) + 47.92 = \$115.87$$

Using the premium/discount formula we find

$$P = C + (Fr - Ci)a_{\overline{n}|} = 105 + (5 - 4.2)a_{\overline{20}|0.04} = \$115.87$$

Using the base amount formula we find

$$P = G + (C - G)\nu^n = 125 + (105 - 125)(1.04)^{-20} = \$115.87$$

Using the Makeham formula we find

$$P = K + \frac{g}{i}(C - K) = 47.92 + \frac{100}{84}(105 - 47.92) = \$115.87 \blacksquare$$

Example 43.2
For the bond of the previous example, determine the following:
(a) Nominal yield based on the par value.
(b) Nominal yield, based on the redemption value.
(c) Current yield.
(d) Yield to maturity.

Solution.
(a) The nominal yield is the annualized coupon value to the par value,i.e. $\frac{2Fr}{F} = \frac{10}{100} = 10\%$.
(b) This is the annualized modified coupon rate

$$2g = 2\frac{Fr}{C} = 2\frac{5}{105} = 9.52\%$$

(c) Current yield is the ratio of annualized coupon to price

$$\frac{10}{115.87} = 8.63\%$$

(d) Yield to maturity is the annualized actual yield which is 8% $\blacksquare$

In the problems considered thus far, we have assumed that coupon payments are constant. It is possbile to have bonds with varying coupon payments. In this case, the coupons will constitute a varying annuity which can be evaluated using the approach of Sections 26 - 27. Again, the price of the bond is the present value of all future coupon payments plus the present value of the redemption value.

Example 43.3
A 1000 par value 20 year bond with annual coupons and redeemable at maturity at 1050 is purchased for P to yield an annual effective rate of 8.25%. The first coupon is 75. Each subsequent coupon is 3% greater than the preceding coupon.
Determine P.

Solution.
The coupon payments constitute a annuity that is varying in geometric progression. The present value of this annuity is

$$75\frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k} = 75\frac{1-\left(\frac{1.03}{1.0825}\right)^{20}}{0.0825-0.03} = 900.02.$$

The present value of the redemption value is

$$1050(1.0825)^{-20} = 215.09.$$

Thus, the price of the bond is

$$900.02 + 215.09 = 1115.11 \blacksquare$$

It is possible that the bond yield rate is not constant over the term of the bond. We illustrate this in the following example.

Example 43.4
Find the price of a 1000 par value 10-year bond with coupons of 8.4% convertible semi-annually, which will be redeemed at 1050. The bond yield rate is 10% convertible semi-annually for the first five years and 9% convertible semi-annually for the second five years.

Solution.
The present value of all future coupons is

$$42[a_{\overline{10}|0.05} + (1.05)^{-10}a_{\overline{10}|0.045}] = 528.33.$$

The present value of the redemption value is

$$1050(1.05)^{-10}(1.045)^{-10} = 415.08.$$

Thus, the price of the bond is

$$528.33 + 415.08 = 943.41 \blacksquare$$

Next, we consider cases when the yield rate and coupon rate are at different frequencies. Consider first the case in which each coupon period contains k yield rate conversion periods. Suppose the total number of yield rate conversion periods over the term of the bond is n. There is a coupon payment of Fr paid at the end of each k yield rate conversion periods. The present value of all future coupon payments is $Fr\frac{a_{\overline{n}|}}{s_{\overline{k}|}}$ and the present value of the redemption value is Cv^n. Thus, the price of the bond is

$$P = Fr\frac{a_{\overline{n}|}}{s_{\overline{k}|}} + Cv^n.$$

Example 43.5
Find the price of a $1000 par value 10-year bond maturing at par which has coupons at 8% convertible semi-annually and is bought to yield 6% convertible quarterly.

Solution.
The price of the bond is

$$P = 40\frac{a_{\overline{40}|0.015}}{s_{\overline{2}|0.015}} + 1000(1.015)^{-40} = 1145.12 \blacksquare$$

The second case is when each yield rate conversion period contains m coupon conversion periods. Again, we let n be the total number of yield rate conversion periods. There is a coupon of $\frac{Fr}{m}$ paid at the end of each mth of a conversion period. Thus, the present value of the future coupon payments is $Fra_{\overline{n}|}^{(m)}$ and the present value of the redemption value is $C\nu^n$. Thus, the price of the bond is

$$P = Fra_{\overline{n}|}^{(m)} + C\nu^n.$$

Example 43.6
Find the price of a $1000 par value 10-year bond maturing at par which has coupons at 8% convertible quarterly and is bought to yield 6% convertible semi-annually.

Solution.
The price of the bond is

$$\begin{aligned}P =&40a_{\overline{20}|}^{(2)} + 1000(1.03)^{-20}\\ =&40\frac{1-1.03^{-20}}{2[(1.03)^{\frac{1}{2}}-1]} + 1000(1.03)^{-20}\\ =&1153.21\blacksquare\end{aligned}$$

Practice Problems

Problem 43.1
A 10-year $1,000 par value bond bearing a 8.4% coupon rate convertible semiannually and redeemable at $1,050 is bought to yield 10% convertible semiannually. Find the price. Use all four formulas.

Problem 43.2
For the bond of the previous problem, determine the following:
(a) Nominal yield based on the par value.
(b) Nominal yield, based on the redemption value.
(c) Current yield.
(d) Yield to maturity.

Problem 43.3
A 5-year $100 par value bond bearing a 8% coupon rate payable semi-annually is selling at par value. If prevailing market rates of interest suddenly go to 10% convertible semiannually, find the percentage change in the price of the bond.

Problem 43.4
Two 1000 bonds redeemable at par at the end of the same period are bought to yield 4% convertible semiannually. One bond costs $1136.78, and has a coupon rate of 5% payable semiannually. The other bond has a coupon rate of 2.5% payable semiannually. Find the price of the second bond.

Problem 43.5
A $n-$year $1000 par value bond bearing a 9% coupon rate payable semiannually and redeemable at $1125 is bought to yield 10% convertible semiannually. Find the purchase price of the bond if the present value of the redemption value is $225 at the given yield rate.

Problem 43.6
A $n-$year $1000 par value bond maturing at par and with semi-annual coupon of $50 is purchased for $1110. Find the base amount G if the redemption value is $450.

Problem 43.7
An investor owns a 1000 par value 10% bond with semiannual coupons. The bond will mature at par at the end of 10 years. The investor decides that an 8-year bond would be preferable. Current yield rates are 7% convertible semiannually. The investor uses the proceeds from the sale of the 10% bond to purchase a 6% bond with semiannual coupons, maturing at par at the end of 8 years. Find the par value of the 8-year bond.

Problem 43.8
An $n-$year 1000 par value bond matures at par and has a coupon rate of 12% convertible semiannually. It is bought at a price to yield 10% convertible semiannually. If the term of the bound is doubled, the price will increase by 50. Find the price of the $n-$year bond.

Problem 43.9
A 20 year bond with a par value of \$10,000 will mature in 20 years for \$10,500. The coupon rate is 8% convertible semiannually. Calculate the price that Andrew would pay if he bought the bond to yield 6% convertible twice a year.

Problem 43.10
After 5 years (immediately after the 10th coupon is paid), Andrew decides to sell the bond in the previous problem. Interest rates have changed such that the price of the bond at the time of the sale is priced using a yield rate of 9% convertible semiannually. Calculate the selling price.

Problem 43.11
A bond matures for its par value of 1000 in one year. The bond pays a annual coupon on 80 each year. The price of the bond is 800.
Which of the following are true?
i. The nominal yield is 8% per annum.
ii. The current yield is 10% per annum.
iii. The yield to maturity is 35% per annum

Problem 43.12
You paid \$950 for a 7 year bond with a face value of \$1,000 and semi-annual coupons of \$25. What is your current yield on this bond and what is your nominal annual yield to maturity (convertible semi-annually)?

Problem 43.13
A 15-year bond with a coupon of \$X payable every 6 months has a face (and redemption) value of \$10,000. At the nominal annual interest rate, convertible semi-annually, of 6.5%, the price of the bond is \$8,576.36. What is X?

Problem 43.14 ‡
A ten-year 100 par value bond pays 8% coupons semiannually. The bond is priced at 118.20 to yield an annual nominal rate of 6% convertible semiannually. Calculate the redemption value of the bond.

Problem 43.15 ‡
Susan can buy a zero coupon bond that will pay 1000 at the end of 12 years and is currently selling for 624.60 . Instead she purchases a 6% bond with coupons payable semi-annually that will pay 1000 at the end of 10 years. If she pays X she will earn the same annual effective interest rate as the zero coupon bond. Calculate X.

Problem 43.16 ‡
A 30-year bond with a par value of 1000 and 12% coupons payable quarterly is selling at 850. Calculate the annual nominal yield rate convertible quarterly.

Problem 43.17 ‡
An investor borrows an amount at an annual effective interest rate of 5% and will repay all interest and principal in a lump sum at the end of 10 years. She uses the amount borrowed to purchase a 1000 par value 10-year bond with 8% semiannual coupons bought to yield 6% convertible semiannually. All coupon payments are reinvested at a nominal rate of 4% convertible semiannually. Calculate the net gain to the investor at the end of 10 years after the loan is repaid.

Problem 43.18 ‡
Dan purchases a 1000 par value 10-year bond with 9% semiannual coupons for 925. He is able to reinvest his coupon payments at a nominal rate of 7% convertible semiannually. Calculate his nominal annual yield rate convertible semiannually over the ten-year period.

Problem 43.19 ‡
You have decided to invest in Bond X, an $n-$year bond with semi-annual coupons and the following characteristics:
• Par value is 1000.
• The ratio of the semi-annual coupon rate to the desired semi-annual yield rate, $\frac{r}{i}$, is 1.03125.
• The present value of the redemption value is 381.50.
Given $v^n = 0.5889$, what is the price of bond X?

Problem 43.20 ‡
Bill buys a 10-year 1000 par value 6% bond with semi-annual coupons. The price assumes a nominal yield of 6%, compounded semi-annually. As Bill receives each coupon payment, he immediately puts the money into an account earning interest at an annual effective rate of i. At the end of 10 years, immediately after Bill receives the final coupon payment and the redemption value of the bond, Bill has earned an annual effective yield of 7% on his investment in the bond. Calculate i.

Problem 43.21
Two \$1000 par value bonds redeemable at par at the end of the same period are bought to yield

4% convertible quarterly. One bond costs \$1098 and has a coupon rate of 5% payable quarterly. The other bond has a coupon rate of 3% payable quarterly. Find the price of the second bond to the nearest dollar.

Problem 43.22
A 1000 par value $n-$year bond maturing at par with annual coupons of 100 is purchased for 1125. The present value of the redemption value is 500. Find n to the nearest integer.

Problem 43.23
A 1000 bond with annual coupons is redeemable at par at the end of 10 years. At a purchase price of 870, the yield rate is i. The coupon rate is $i - 0.02$. Calculate i.

Problem 43.24
A bond with coupons equal to 40 sells for P. A second bond with the same maturity value and term has coupons equal to 30 and sells for Q. A third bond with the same maturity value and term has coupons equal to 80. All prices are based on the same yield rate, and all coupons are paid at the same frequency. Determine the price of the third bond in terms of P and Q.

Problem 43.25 ‡
You have decided to invest in two bonds. Bond X is an $n-$year bond with semi-annual coupons, while bond Y is an accumulation bond redeemable in $\frac{n}{2}$ years. The desired yield rate is the same for both bonds. You also have the following information:
Bond X
- Par value is 1000.
- The ratio of the semi-annual bond rate to the desired semi-annual yield rate, $\frac{r}{i}$ is 1.03125.
- The present value of the redemption value is 381.50.

Bond Y
- Redemption value is the same as the redemption value of bond X.
- Price to yield is 647.80.

What is the price of bond X?

Problem 43.26
You are given two $n-$year par value 1,000 bonds. Bond X has 14% semi-annual coupons and a price of 1,407.70 to yield i compounded semi-annually. Bond Y has 12% semi-annual coupons and a price of 1,271.80 to yield the same rate i compounded semi-annually. Calculate the price of Bond X to yield $i - 1\%$.

Problem 43.27
Henry has a five year 1,000,000 bond with coupons at 6% convertible semi-annually. Jean buys a 10 year bond with face amount X and coupons at 6% convertible semi-annually. Both bonds are redeemable at par. Henry and Jean both buy their bonds to yield 4% compounded semi-annually and immediately sell them to an investor to yield 2% compounded semi-annually. Jean earns the same amount of profit as Henry. Calculate X.

Problem 43.28
On June 1, 1990, an investor buys three 14 year bonds, each with par value 1000, to yield an effective annual interest rate of i on each bond. Each bond is redeemable at par. You are given
(1) the first bond is an accumulation bond priced at 195.63
(2) the second bond has 9.4% semiannual coupons and is priced at 825.72
(3) the third bond has 10% annual coupons and is priced at P.
Calculate P.

Problem 43.29
A 10 year bond with par value 1000 and annual coupon rate r is redeemable at 1100. You are given
(1) the price to yield an effective annual interest rate of 4% is P
(2) the price to yield an effective annual interest rate of 5% is $P - 81.49$
(3) the price to yield an effective annual interest rate of r is X.
Calculate X.

Problem 43.30
Patrick buys a 28 year bond with a par value of 1200 and annual coupons. The bond is redeemable at par. Patrick pays 1968 for the bond, assuming an annual effective yield rate of i. The coupon rate on the bond is twice the yield rate. At the end of 7 years Patrick sells the bond for P, which produces the same annual effective yield rate of i to the new buyer. Calculate P.

Problem 43.31
A 10 year bond with coupons at 8% convertible quarterly will be redeemed at 1600. The bond is bought to yield 12% convertible quarterly. The purchase price is 860.40. Calculate the par value.

Problem 43.32
A corporation decides to issue an inflation-adjusted bond with a par value of \$1000 and with annual coupons at the end of each year for 10 years. The initial coupon rate is 7% and each coupon is 3% greater than the preceding coupon. The bond is redeemed for \$1200 at the end of 10 years. Find the price an investor should pay to produce a yield rate of 9% effective.

Problem 43.33
A 1000 par value 10 year bond with semiannual coupons and redeemable at 1100 is purchased at 1135 to yield 12% convertible semiannually. The first coupon is X. Each subsequent coupon is 4% greater than the preceding coupon. Determine X.

Problem 43.34
A 1,000 par value 3 year bond with annual coupons of 50 for the first year, 70 for the second year, and 90 for the third year is bought to yield a force of interest

$$\delta_t = \frac{2t-1}{2(t^2-t+1)}, \quad t \geq 0.$$

Calculate the price of this bond.

44 Amortization of Premium or Discount

Bonds can be priced at a premium, discount, or at par. If the bond's price is higher than its par value,i.e. $P > C$, then the bond is said to sell at a **premium** and the difference $P - C$ is called the "premium". On the other hand, if the bond's price is less than its par value,i.e. $P < C$, then the bond is said to sell at a **discount** and the difference $C - P$ is called the "discount". Thus, a discount is merely a negative premium.
Using the premium/discount formula of the previous section we see that

$$\text{Premium} = P - C = (Fr - Ci)a_{\overline{n}|i} = C(g - i)a_{\overline{n}|i} \text{ if } g > i$$

In this case, a loss in that amount incurred when the bond is redeemed. Similarly,

$$\text{Discount} = C - P = (Ci - Fr)a_{\overline{n}|i} = C(i - g)a_{\overline{n}|i} \text{ if } g < i$$

and a gain in that amount is incurred when the bond is redeemed.
Because of this profit or loss at redemption, the amount of each coupon CANNOT be considered as interest income to an investor. It is necessary to divide each coupon into interest earned and principal adjustment portions similar to the separation of payments into interest and principal when discussing loans.
In the same sense that a loan has an outstanding balance at any time, we can talk of the book value of a bond at any time t. We define the **book value** to be the present value of all future payments. If we let B_t be the book value after the tth coupon has just been paid, then the value of the remaining payments are:$n - t$ coupons and a payment of C at the date of redemption. Hence,

$$B_t = Fra_{\overline{n-t}|} + C\nu^{n-t}.$$

Example 44.1
Find the book value immediately after the payment of the 14th coupon of a 10-year 1,000 par value bond with semiannual coupons, if $r = 0.05$ and the yield rate is 12% convertible semiannually.

Solution.
We have $F = C = 1000, n = 20, r = 0.05$, and $i = 0.06$. Thus,

$$B_{14} = 1000(0.05)a_{\overline{6}|0.06} + 1000(1.06)^{-6} = 950.83 \blacksquare$$

Letting $t = 0$ in the expression of B_t we find $B_0 = P$. Similarly, $B_n = C$. Thus, on the redemption date, the book value of a bond equals the redemption value of the bond.
From this it follows that when a bond is purchased at a premium, the book value of the bond will be written -down(decreased) at each coupon date, so that at the time of redemption its book value

equals the redemption value. This process is called **amortization of premium**.
When it is puchased at a discount, the book value of the bond will be written-up(increased) at each coupon date, so that at the time of redemption its book value equals the redemption value. This process is called **accumulation of discount**.
Book values are very useful in constructing bond amortization schedules. A **bond amortization schedule** shows the division of each bond coupon into interest earned and principal adjustment portions, together with the book value after each coupon is paid.
We will denote the interest earned after the tth coupon has been made by I_t and the corresponding principal adjustment portion by P_t. The level coupon will continue to be denoted by Fr. Note that

$$\begin{aligned}I_t =&iB_{t-1} = i[Fra_{\overline{n-t+1}|} + C\nu^{n-t+1}]\\ =&Cg(1-\nu^{n-t+1}) + iC\nu^{n-t+1} = Cg + C(i-g)\nu^{n-t+1}\end{aligned}$$

and

$$P_t = Fr - I_t = Cg - I_t = C(g-i)\nu^{n-t+1}.$$

where P_t is positive in the case of a premium and negative in the case of a discount. It is easy to see that

$$B_t = B_0 - \sum_{k=1}^{t} P_k.$$

Example 44.2
A \$1000 bond, redeemable at par on December 1, 1998, with 9% coupons paid semiannually. The bond is bought on June 1, 1996. Find the purchase price and construct a bond amortization schedule if the desired yield is 8% compounded semiannually.

Solution.
Using the premium/discount formula with $C = 1000, F = 1000, r = 4.5\%$, and $i = 4\%$, the purchase price on June 1, 1996 is

$$P = 1000 + (45 - 40)a_{\overline{5}|0.04} = \$1022.26.$$

Thus, the bond is purchased at a premium of \$22.26.
For the bond amortization schedule, we first do the calculation of the first line of the schedule. Each successive line of the schedule is calculated in a similar fashion. The interest earned portion of the first coupon, i.e. on Dec 1, 1996 is

$$I_1 = iB_0 = 0.04(1022.26) = 40.89.$$

The principal adjustment portion of the first coupon is

$$P_1 = Fr - I_1 = 45 - 40.89 = \$4.11.$$

The book value at the end of the first period is

$$B_1 = B_0 - P_1 = \$1018.15.$$

Date	Coupon	Interest earned	Amount for amortization of premium	Book value
June 1, 1996	0	0	0	1022.26
Dec 1, 1996	45.00	40.89	4.11	1018.15
June 1, 1997	45.00	40.73	4.27	1013.88
Dec 1, 1997	45.00	40.56	4.44	1009.44
June 1, 1998	45.00	40.38	4.62	1004.82
Dec 1, 1998	45.00	40.19	4.81	1000.01
Total	225.00	202.74	22.25	

The one cent error is due to rounding ■

The following observations can be concluded from the above amortization schedule:
(1) The sum of the principal adjustment column is equal to the amount of premium/discount.
(2) The sum of the interest paid column is equal to the difference between the sum of the coupons and the sum of the principal adjustment column.
(3) The principal adjustment column is a geometric progression with common ration $1 + i$.

The following is an example of a bond bought at discount.

Example 44.3
A \$1000 bond, redeemable at par on December 1, 1998, with 9% coupons paid semiannually. The bond is bought on June 1, 1996. Find the purchase price and construct a bond amortization schedule if the desired yield is 10% compounded semiannually.

Solution.
Using the premium/discount formula with $C = 1000, F = 1000, r = 4.5\%$, and $i = 5\%$, the purchase price on June 1, 1996 is

$$P = 1000 + (45 - 50)a_{\overline{5}|0.05} = \$978.35.$$

Thus, the bond is purchased at a discount of \$21.65.

Date	Coupon	Interest earned	Amount for accumulation of discount	Book value
June 1, 1996	0	0	0	978.35
Dec 1, 1996	45.00	48.92	−3.92	982.27
June 1, 1997	45.00	49.11	−4.11	986.38
Dec 1, 1997	45.00	49.32	−4.32	990.70
June 1, 1998	45.00	49.54	−4.54	995.24
Dec 1, 1998	45.00	49.76	−4.76	1000.00
Total	225.00	246.65	−21.65	

Much like dealing with loans, if it is desired to find the interest earned or principal adjustment portion of any one coupon, it is not necessary to construct the entire table. Simply find the book value at the beginning of the period in question which is equal to the price at that point computed at the original yield rate and can be determined by the methods of Section 41. Then find that one line of the table.

Example 44.4
Consider a \$1000 par value two-year 8% bond with semiannual coupons bought to yield 6% convertible semiannually. Compute the interest portion earned and the principal adjustment portion of the 3rd coupon payment.

Solution.
We have $F = C = 1000, n = 4, r = 0.04$, and $i = 0.03$. Thus, the coupon payment is $Fr = \$40.00$. The principal adjustment portion of the 3rd coupon payment is $P_3 = B_3 - B_2 = 40a_{\overline{1}|0.03} + 1000(1.03)^{-1} - (40a_{\overline{2}|0.03} + 1000(1.03)^{-2}) = \$9.43$. The interest earned portion is $I_3 = 40.00 - 9.43 = \$30.57$ ■

Another method of writing up or writing down the book values of bonds is the straight line method. This does not produce results consistent with compound interest, but it is very simple to apply. In this method, the book values are linear, grading from $B_0 = P$ to $B_n = C$. Thus, the principal adjustment column is constant and is given by

$$P_t = \frac{P - C}{n}, \quad t = 1, 2, \cdots, n$$

and the interest earned portion is

$$I_t = Fr - P_t, \quad t = 1, 2, \cdots, n$$

Note that $P_t > 0$ for premium bond and $P_t < 0$ for discount bond.

Example 44.5
Find the book values for the bond in Example 44.2 by the straight line method.

Solution.
We have $P_t = \frac{22.25}{5} = 4.45$. Thus,

$$\begin{aligned}
B_0 =& 1022.26\\
B_1 =& 1022.26 - 4.45 = 1017.81\\
B_2 =& 1017.81 - 4.45 = 1013.36\\
B_3 =& 1013.36 - 4.45 = 1008.91\\
B_4 =& 1008.91 - 4.45 = 1004.46\\
B_5 =& 1004.46 - 4.45 = 1000.01 \blacksquare
\end{aligned}$$

Practice Problems

Problem 44.1
A \$1000 par value two-year 8% bond with semiannual coupons is bought to yield 6% convertible semiannually.
(a) Create an amortization schedule for this transaction.
(b) Find the book values of the bond using the straight line method.

Problem 44.2
A \$1000 par value two-year 8% bond with semiannual coupons is bought to yield 10% convertible semiannually.
(a) Create an amortization schedule for this transaction.
(b) Find the book values of the bond using the straight line method.

Problem 44.3
For a bond of face value 1 the coupon rate is 150% of the yield rate, and the premium is p. For another bond of 1 with the same number of coupons and the same yield rate, the coupon rate is 75% of the yield rate. Find the price of the second bond.

Problem 44.4
For a certain period a bond amortization schedule shows that the amount for amortization of premium is 5, and that the required interest is 75% of the coupon. Find the amount of the coupon.

Problem 44.5
A 10-year bond with semi-annual coupons is bought at a discount to yield 9% convertible semiannually. If the amount for accumulation of discount in the next-to-last coupon is 8, find the total amount for accumulation of discount during the first four years in the bond amortization schedule.

Problem 44.6
A 1000 par value 5-year bond with a coupon rate of 10% payable semiannually and redeemable at par is bought to yield 12% convertible semiannually. Find the total of the interest paid column in the bond amortization schedule.

Problem 44.7
A five year bond with a par value of 1000 will mature in 5 years for 1000. Annual coupons are payable at a rate of 6%. Create the bond amortization schedule if the bond is bought to yield 8% annually.

Problem 44.8
A 40 year bond with a par value of 5000 is redeemable at par and pays semi-annual coupons at a rate of 7% convertible semi-annually. The bond is purchased to yield annual effective rate of 6%. Calculate the amortization of the premium in the 61st coupon.

Problem 44.9
A 10 year bond is redeemable at par of 100,000. The bond pays semi-annual coupons of 4000. The bond is bought to yield 6% convertible semi-annually. Calculate the premium paid for the bond.

Problem 44.10
A bond is redeemable in eight years at 120% of its par value. The bond pays an annual coupon rate of 6%. Calculate the premium as a percent of the par value if the bond is purchased to yield 6% annually.

Problem 44.11
Consider a three-year bond, with a $1,000 face value and a 9% coupon rate paid semi-annually, which was bought to yield 7% convertible semi-annually. Find the amount for amortization of premium during the bonds third half-year (i.e., between the second and third coupon payments).

Problem 44.12
A bond of face amount 100 pays semi-annual coupons and is purchased at a premium of 36 to yield annual interest of 7% compounded semiannually. The amount for amortization of premium in the 5th coupon is 1.00. What is the term of the bond?

Problem 44.13 ‡
Among a company's assets and accounting records, an actuary finds a 15-year bond that was purchased at a premium. From the records, the actuary has determined the following:
(i) The bond pays semi-annual interest.
(ii) The amount for amortization of the premium in the 2nd coupon payment was 977.19.
(iii) The amount for amortization of the premium in the 4th coupon payment was 1046.79.
What is the value of the premium?

Problem 44.14 ‡
A 10,000 par value 10-year bond with 8% annual coupons is bought at a premium to yield an annual effective rate of 6%. Calculate the interest portion of the 7th coupon.

Problem 44.15 ‡
A 1000 par value 5 year bond with 8% semiannual coupons was bought to yield 7.5% convertible semiannually. Determine the amount of premium amortized in the sixth coupon payment.

Problem 44.16
On May 1, 1985, a bond with par value 1000 and annual coupons at 5.375% was purchased to yield an effective annual interest rate of 5%. On May 1, 2000, the bond is redeemable at 1100. The book value of the bond is adjusted each year so that it equals the redemption value on May 1, 2000. Calculate the amount of write-up or write-down in the book value in the year ending May 1, 1991.

Problem 44.17
A bond with par value of 1000 and 6% semiannual coupons is redeemable for 1100. You are given
(1) the bond is purchased at P to yield 8% convertible semiannually and
(2) the amount of principal adjustment for the 16th semiannual period is 5
Calculate P.

Problem 44.18
Laura buys two bonds at time 0.
Bond X is a 1000 par value 14 year bond with 10% annual coupons. It is bought at a price to yield an annual effective rate of 8%.
Bond Y is a 14 year par value bond with 6.75% annual coupons and a face amount of F.
Laura pays P for the bond to yield an annual effective rate of 8%. During year 6 the writedown in premium (principal adjustment) on bond X is equal to the writeup in discount (principal adjustment) on bond Y. Calculate P.

Problem 44.19
A 1000 par value 18 year bond with annual coupons is bought to yield an annual effective rate of 5%. The amount for amortization of premium in the 10th year is 20. The book value of the bond at the end of year 10 is X. Calculate X.

Problem 44.20
Thomas buys a bond at a premium of 200 to yield 6% annually. The bond pays annual coupons and is redeemable for its par value of 1000. Calculate the amount of interest in the first coupon.

Problem 44.21
Megan buys a bond that is redeemable for its par value of 20,000 after 5 years. The bond pays coupons of 800 annually. The bond is bought to yield 8% annually. Calculate the accumulation of discount in the 4th coupon.

Problem 44.22
Jenna buys a bond at a premium. The principal amortized in the first annual coupon is half that amortized in the 11th annual coupon. Calculate the yield rate used to calculate the purchase price of the bond.

Problem 44.23
Betty buys an $n-$year 1000 par value bond with 6.5% annual coupons at a price of 798.48. The price assumes an annual effective yield rate of i. The total write-up in book value of the bond during the first 3 years after purchase is 22.50.

Problem 44.24
Ann purchases a 1000 face value 20-year bond, redeemable at 1200, with 5% annual coupons, at a price that yields her an annual effective interest rate of 3%. Immediately upon receiving each coupon, she invests the coupon in an account that earns an annual effective interest rate of 7%. Immediately after receiving the eighth coupon, Ann sells the bond at a price that yields the new purchaser an annual effective interest rate of 4.75%. Determine Ann's yield rate, as an annual effective interest rate, over the 8-year period that she owns the bond.

Problem 44.25
A 1000 face value 20-year 8% bond with semiannual coupons is purchased for 1014. The redemption value is 1000. The coupons are reinvested at a nominal annual rate of 6%, compounded semiannually. Determine the purchaser's annual effective yield rate over the 20- year period.

Problem 44.26
An $n-$year 1000 par value bond with 8% annual coupons has an annual effective yield of $i, i > 0$. The book value of the bond at the end of year 3 is 1099.84 and the book value at the end of year 5 is 1082.27. Calculate the purchase price of the bond.

Problem 44.27
Bryan buys a $2n-$year 1000 par value bond with 7.2% annual coupons at a price P. The price assumes an annual effective yield of 12%. At the end of n years, the book value of the bond, X, is 45.24 greater than the purchase price, P. Assume $\nu_{0.12}^n < 0.5$. Calculate X.

45 Valuation of Bonds Between Coupons Payment Dates

Up to now, bond prices and book values have been calculated assuming the coupon has just been paid. In particular, we can find a relationship between the book values of two consecutive coupon dates t and $t+1$ as follows.
Let B_t and B_{t+1} be the book values just after the tth and $(t+1)$th coupons are paid and Fr the amount of the coupon. Then

$$\begin{aligned}
B_t(1+i) - Fr =& Fr\left(\frac{1-\nu^{n-t}}{i}\right)(1+i) + C\nu^{n-t}(1+i) - Fr\\
=&\frac{Fr}{i} - \frac{Fr\nu^{n-t}}{i}(1+i) + C\nu^{n-t-1}\\
=&\frac{Fr}{i} - \frac{Fr\nu^{n-t-1}}{i} + C\nu^{n-t-1}\\
=&Fr\left(\frac{1-\nu^{n-t-1}}{i}\right) + C\nu^{n-t-1}\\
=&Fra_{\overline{n-t-1}|} + C\nu^{n-t-1}\\
=&B_{t+1}
\end{aligned}$$

where we assume a constant yield rate i over the interval.

Example 45.1
You buy an $n-$year 1,000 par value bond with 6.5% annual coupons at a price of 825.44, assuming an annual yield rate of $i, i > 0$.
After the first two years, the bond's book value has changed by 23.76. Calculate i.

Solution.
We have $B_2 - B_0 = B_0(1+i)^2 - Fr(1+i) - Fr - B_0$. This implies $23.76 = 825.44(1+i)^2 - 65(1+i) - 65 - 825.44$ which reduces to

$$825.44(1+i)^2 - 65(1+i) - 914.20 = 0$$

Solving this with the quadratic formula we find $1+i = 1.0925 \rightarrow i = 9.25\%$ ■

Book values can be computed at times other than immediately after a coupon payment. In this section, we would like to analyze the book value B_{t+k} for $0 < k < 1$ where time k is measured from the last coupon payment.
When buying an existing bond between its coupon dates, one must decide how to split up the

coupon between the prior owner and the new owner. Consider the price that a buyer would pay for the bond a fractional time k through a coupon period. Assume that the buyer will obtain a yield rate equal to that of the current bond holder. The buyer will receive all of the next coupon. The current holder would expect to receive part of this coupon as interest for the period which we call **accrued interest** or **accrued coupon** and is denoted by Fr_k. Note that $Fr_0 = 0$ and $Fr_1 = Fr$, the coupon payment, where Fr_1 is computed just before the coupon is paid.
How should the purchase price be allocated between accrued interest (or accrued coupon) and price for the bond? The purchase price for the bond is called the **flat price.** It is defined to be the money which actually changes hands at the date of sale. It is denoted by B^f_{t+k}. The price for the bond is the book value, which is also called the **market price** and is denoted by B^m_{t+k}. The market price is the price commonly quoted in the financial press, since the market price changes smoothly through time, while the flat price fluctuates due to coupon accrual. From the above definitions, it is clear that

$$B^f_{t+k} = B^m_{t+k} + Fr_k.$$

Example 45.2
Consider the amortization schedule of Example 44.2.

Date	Coupon	Interest earned	Amount for amortization of premium	Book value
June 1, 1996	0	0	0	1022.26
Dec 1, 1996	45.00	40.89	4.11	1018.15
June 1, 1997	45.00	40.73	4.27	1013.88
Dec 1, 1997	45.00	40.56	4.44	1009.44
June 1, 1998	45.00	40.38	4.62	1004.82
Dec 1, 1998	45.00	40.19	4.81	1000.01
Total	225.00	202.74	22.25	

Sketch a graph describing the relationship between the flat price and the market price.

Solution.
The graph is shown in Figure 45.1. Note that the accrued coupon at any date is equal to the vertical distance between the solid line and the dotted line ■

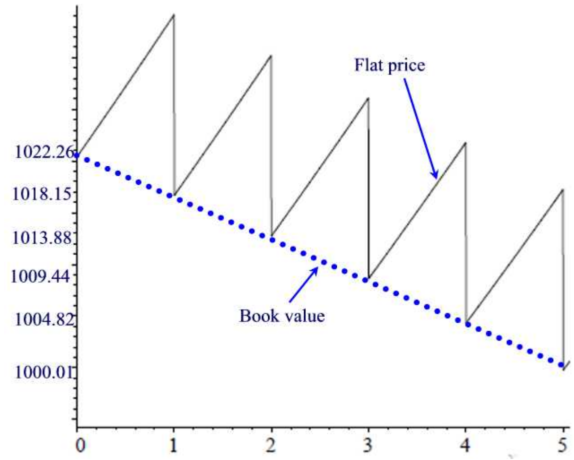


Figure 45.1

There are three methods used to find the flat price, the market price, and the accured coupon. Values on the coupon dates are known, so the differences among the three methods arise only for the interim values between coupon dates.
First, we have the **theoretical method**. This is an exact method, based on compound interest. This method argues that the flat price should be the book value B_t after the preceding coupon, accumulated by $(1+i)^k$, where i is the yield rate, giving the flat price of

$$B^f_{t+k} = (1+i)^k B_t.$$

The accrued coupon is given by (See Problem 15.48)

$$Fr_k = Fr\left[\frac{(1+i)^k - 1}{i}\right].$$

The market value is then the difference

$$B^m_{t+k} = (1+i)^k B_t - Fr\left[\frac{(1+i)^k - 1}{i}\right]$$

The second method is called the **practical method.** It is an approximate method that uses the linear approximation $(1+i)^k \approx 1 + ki$ for $0 < k < 1$. Thus, obtaining

$$\begin{aligned}
B^f_{t+k} =&(1+ki)B_t\\
Fr_k =&kFr\\
B^m_{t+k} =&(1+ki)B_t - kFr = B_t + kiB_t - kFr\\
=&B_t + k[B_{t+1} + Fr - B_t] - kFr = (1-k)B_t + kB_{t+1}
\end{aligned}$$

where we used the formula

$$B_t(1+i) - Fr = B_{t+1}.$$

The third method called the **semi-theoretical method** is the most widely used method, and has been accepted as the standard method of calculation by the securities industry. The flat price is determined as in the theoretical method, and the accrued coupon is determined as in the practical method.

$$\begin{aligned} B^f_{t+k} =&(1+i)^k B_t \\ Fr_k =&kFr \\ B^m_{t+k} =&(1+i)^k B_t - kFr \end{aligned}$$

In computing k, we use an actual count of days:

$$k = \frac{\text{number of days since last coupon paid}}{\text{number of days in the coupon period}}.$$

Both the actual/actual and 30/360 methods (see Section 5) can be used. See Problem 45.12.

Example 45.3
Consider a \$1000 par value two-year 8% bond with semiannual coupons bought to yield 6% convertible semiannually. The price of the bond is computed to be \$1037.17. Compute the flat price, accrued interest, and market price five months after purchase of the bond.

Solution.
We have $Fr = 1000(0.04) = \$40.00$ and $k = \frac{5}{6}$. With the theoretical method we find

$$\begin{aligned} B^f_{\frac{5}{6}} =&1037.17(1.03)^{\frac{5}{6}} = \$1063.04 \\ Fr_{\frac{5}{6}} =&40\left[\frac{(1.03)^{\frac{5}{6}} - 1}{0.03}\right] = \$33.25 \\ B^m_{\frac{5}{6}} =&1063.04 - 33.25 = \$1029.79 \end{aligned}$$

With the practical method we find

$$\begin{aligned} B^f_{\frac{5}{6}} =&1037.17\left[1 + \frac{5}{6}(0.03)\right] = \$1063.10 \\ Fr_{\frac{5}{6}} =&\frac{5}{6}(40) = \$33.33 \\ B^m_{\frac{5}{6}} =&1063.10 - 33.33 = \$1029.77 \end{aligned}$$

Finally, with the semi-theoretical method we have

$$\begin{aligned} B^f_{\frac{5}{6}} =& 1037.17(1.03)^{\frac{5}{6}} = \$1063.04 \\ Fr_{\frac{5}{6}} =& \frac{5}{6}(40) = \$33.33 \\ B^m_{\frac{5}{6}} =& 1063.04 - 33.33 = \$1029.71 \blacksquare \end{aligned}$$

Example 45.4
A 1000 par value 5 year bond with semi-annual coupons of 60 is purchased to yield 8% convertible semi-annually. Two years and two months after purchase, the bond is sold at the flat price which maintains the yield over the two years and two months. Calculate the flat price using the theoretical method.

Solution.
The price of the bond after two years is

$$P = 60a_{\overline{6}|0.04} + 1000\nu^{-6}_{0.04} = 1,104.84.$$

The flat price is

$$B^f_{\frac{2}{6}} = 1104.84(1.04)^{\frac{2}{6}} = 1,119.38 \blacksquare$$

We conclude this section by finding the premium or discount between coupon payment dates. For the premium we have the formula

$$\text{Premium} = B^m_{t+k} - C, \quad i < g$$

and for the discount we have

$$\text{Discount} = C - B^m_{t+k}, \quad i > g.$$

Thus, premiums and discounts are based on the market value or the book value and not on the flat price.

Example 45.5
A $n-$year \$1000 par value bond is selling at \$1010 and has a market value of \$980 and acrrued coupon of \$30. Find premium/discount amount.

Solution.
Since the premium/discount calculation is based on market value rather than flat price, the bond is a discount bond. The discount amount is \$20 ■

Practice Problems

Problem 45.1
A 10 year bond with semi-annual coupons has a book value immediately after the 5th coupon of 90,000. The flat price 5 months later using the theoretical method is 94,591. Calculate the semi-annual yield on the bond.

Problem 45.2
A bond with semi annual coupons of 2500 has a book value immediately after the 6th coupon of 95,000. The market value using the practical method z months after the 6th coupon is 95,137.50. Calculate z if the yield rate is 7% convertible semi-annually.

Problem 45.3
A 10 year bond with a par value of 100,000 and semi-annual coupons 2500 is bought at a discount to yield 6% convertible semi-annually.
(a) Calculate the book value immediately after the 5th coupon.
(b) Calculate the flat price 4 months after the 5th coupon using the theoretical method.
(c) Calculate the accrued coupon 4 months after the 5th coupon using the theoretical method.
(d) Calculate the market price 4 months after the 5th coupon using the theoretical method.
(e) Calculate (b)-(d) using the practical method.
(f) Calculate (b)-(d) using the semi-theoretical method

Problem 45.4
Which of the following are true for a bond with semi annual coupons?
(I) The flat price of the bond under the theoretical method will always equal the flat price of the bond under the practical method.
(II) The flat price of the bond under the practical method will always exceed the flat price of the bond under the semi-theoretical method.
(III) For a bond purchased at par which matures at par with the yield rate equal to the coupon rate, the flat price at any point in time will equal the par value of the bond under the semi-theoretical method.

Problem 45.5
A bond pays coupons of 600 on April 30 and October 31. Calculate the accrued coupon under the practical method on July 1.

Problem 45.6
A 10 year bond is redeemable at par of 100,000. The bond has semi-annual coupons of 4000. The bond is bought to yield 6% convertible semi-annually. Four months after purchase, calculate the market price based on the theoretical method less the market price based on the semi-theoretical method.

Problem 45.7
A 10-year 100 par value bond bearing a 10% coupon rate payable semi-annually, and redeemable at 105, is bought to yield 8% convertible semiannually. Using the practical method, compute the flat price, accrued interest, and market price three months after purchase of the bond.

Problem 45.8
Show that $B^f_{t+k} = (B_{t+1} + Fr)v^{1-k}$.

Problem 45.9
Arrange in increasing order of magnitude for the three interim bond price methods:
(a) Flat price.
(b) Market price.

Problem 45.10
A 10,000 par value bond with 6% semiannual coupons is being sold 3 years and 2 months before the bond matures. The purchase will yield 8% convertible semiannually to the buyer. Determine the price of the bond. Use the practical method.

Problem 45.11 ‡
A 10,000 par value bond with coupons at 8%, convertible semiannually, is being sold 3 years and 4 months before the bond matures. The purchase will yield 6% convertible semiannually to the buyer. The price at the most recent coupon date, immediately after the coupon payment, was 5640. Calculate the market price of the bond, assuming compound interest throughout.

Problem 45.12
Consider a $1,000 par value bond with coupons at 8% convertible semi-annually and with maturity date 05/06/04. The bond was purchased on 11/06/01. Compute the accrued interest on 12/04/001 taking into account the Actual/Actual day-count basis.

Problem 45.13
A 20-year 10% annual coupon bond has a par value of $1,000. When you originally purchased this bond, the effective annual interest rate was 8%. Suppose that five years after purchase, the effective annual interest rate is 10.5%. What is the absolute difference between the book and market values of the bond at that point in time?

Problem 45.14 ‡
A 1000 bond with semi-annual coupons at $i^{(2)} = 6\%$ matures at par on October 15, 2020. The bond is purchased on June 28, 2005 to yield the investor $i^{(2)} = 7\%$. What is the purchase price?
Assume simple interest between bond coupon dates and note that:

Date	Day of the year
April 15	105
June 28	179
October 15	288

46 Approximation Methods of Bonds' Yield Rates

In this section we consider the question of determining the yield rate to maturity to an investor given the purchase price of a bond. The determination of such yield rates is similar to the determination of an unknown rate of interest for an annuity, discussed in Section 20.

We first consider the yield rate for a bond purchased on a coupon payment date immediately after the coupon is paid. There are two approaches for this problem. One approach is linear interpolation where bond tables are used. We will not pursue this method any further in this book.

Another approach is to develop algebraic estimation formulas for the yield rate. Three main formulas will be discussed. We start with the premium discount formula

$$\begin{aligned} P =& C + (Fr - Ci)a_{\overline{n}|i} \\ =& C + C(g-i)a_{\overline{n}|i} \end{aligned}$$

where F, r, g, C, n, and P are known and i is the unknown. Letting $k = \frac{P-C}{C}$ we find

$$(g-i)a_{\overline{n}|i} = k.$$

Solving this equation for i we obtain

$$i = g - \frac{k}{a_{\overline{n}|i}}$$

which defines i implicitly. We will develop methods for estimating i in this equation.

The first is obtained as follows: Writing the power series expansion of $\frac{1}{a_{\overline{n}|i}}$ (see p.209)

$$\frac{1}{a_{\overline{n}|i}} = \frac{1}{n}\left[1 + \frac{n+1}{2}i + \frac{n^2-1}{12}i^2 + \cdots\right]$$

and neglecting terms of higher than the first degree in i we obtain the estimation

$$i = g - \frac{k}{a_{\overline{n}|i}} \approx g - \frac{k}{n}\left[1 + \frac{n+1}{2}i\right].$$

Solving for i we find

$$i \approx \frac{g - \frac{k}{n}}{1 + \frac{n+1}{2n}k}. \tag{46.1}$$

Formula (46.1) forms the basis of an even simpler method for calculating approximate yield rates called the **bond salesman's method.** Noting that $\frac{n+1}{2n} \approx \frac{1}{2}$ for large n, this method replaces the ratio $\frac{n+1}{2n}$ in formula (46.1) by $\frac{1}{2}$ thus obtaining

$$i \approx \frac{g - \frac{k}{n}}{1 + \frac{1}{2}k}.$$

Error analysis shows that this method produces less accurate results than formula (46.1).
The salesman's formula can be rewritten in the form

$$\begin{aligned} i \approx & \frac{n(Cg)+C-P}{\frac{n}{2}(P+C)} = \frac{n(Fr)+C-P}{\frac{n}{2}(P+C)} \\ = & \frac{\text{Total Interest Paid} + \text{Capital Gain/Loss}}{\text{Number of Periods} \times \text{Average Amount Invested}} \end{aligned}$$

Example 46.1
A \$100 par value 10-year bond with 8% semiannual coupons is selling for \$90. Find the yield rate convertible semi-annually.

Solution.
We have $n = 20, P = 90, C = 100, Fr = 4$ so that by the salesman's formula we have

$$i \approx \frac{n(Fr)+C-P}{\frac{n}{2}(P+C)} = \frac{20(4)+100-90}{0.5(20)(90+100)} = 0.0474$$

so the yield rate is 9.48% convertible semi-annually ■

The third method for estimating i is by solving the equation

$$i = g - \frac{k}{a_{\overline{n}|i}}.$$

by using the Newton-Raphson method. In this case, we are looking for the solutions of the equation $f(i) = 0$ where

$$f(i) = i - g + \frac{k}{a_{\overline{n}|i}} = i - g + \frac{ki}{1-(1+i)^{-n}}.$$

The Newton-Raphson iterations are given by the formula

$$i_{s+1} = i_s - \frac{f(i_s)}{f'(i_s)}.$$

Thus, an expression of the form $i - \frac{f(i)}{f'(i)}$ need to be determined. This is done as follows.

$$\begin{aligned}
i - \frac{f'(i)}{f(i)} =& i - \frac{i - g + \frac{ki}{1-(1+i)^{-n}}}{1 + k\left(\frac{[1-(1+i)^{-n}] - ni(1+i)^{-n-1}}{[1-(1+i)^{-n}]^2}\right)} \\
=& i + \frac{g(1-\nu^n)^2 - i(1-\nu^n)^2 - ki(1-\nu^n)}{(1-\nu^n)^2 + k[(1-\nu^n) - ni\nu^{n+1}]} \\
=& i + \frac{(1-\nu^n)[gia_{\overline{n}|i} - i(1-\nu^n) - ki]}{ia_{\overline{n}|i}(1-\nu^n) + kia_{\overline{n}|i} - nki\nu^{n+1}} \\
=& i + \frac{(1-\nu^n)[ga_{\overline{n}|i} - (1-\nu^n) - \left(\frac{P}{C} - 1\right)]}{a_{\overline{n}|i}(1-\nu^n) + \left(\frac{P}{C} - 1\right) a_{\overline{n}|i} - n\left(\frac{P}{C} - 1\right)\nu^{n+1}} \\
=& i + \frac{(1-\nu^n)\left[ga_{\overline{n}|i} + \nu^n - \frac{P}{C}\right]}{a_{\overline{n}|i}(1-\nu^n) + (g-i)(a_{\overline{n}|i})^2 - n(g-i)a_{\overline{n}|i}\nu^{n+1}} \\
=& i + \frac{ga_{\overline{n}|i} + \nu^n - \frac{P}{C}}{a_{\overline{n}|i} + (g-i)(a_{\overline{n}|i})i^{-1} - n(g-i)i^{-1}\nu^{n+1}} \\
=& i + i\frac{ga_{\overline{n}|i} + \nu^n - \frac{P}{C}}{ia_{\overline{n}|i} + (g-i)(a_{\overline{n}|i}) - n(g-i)\nu^{n+1}} \\
=& i\left[1 + \frac{ga_{\overline{n}|i} + \nu^n - \frac{P}{C}}{a_{\overline{n}|i} + (g-i)(a_{\overline{n}|i}) - n(g-i)\nu^{n+1}}\right] \\
=& i\left[1 + \frac{ga_{\overline{n}|i} + \nu^n - \frac{P}{C}}{ga_{\overline{n}|i} + n(i-g)\nu^{n+1}}\right]
\end{aligned}$$

Thus, the Newton-Raphson iterations are given by

$$i_{s+1} = i_s\left[1 + \frac{ga_{\overline{n}|i_s} + (1+i_s)^{-n} - \frac{P}{C}}{ga_{\overline{n}|i_s} + n(i_s - g)(1+i_s)^{-(n+1)}}\right]$$

An initial starting guess can be chosen to be

$$i_0 = \frac{g - \frac{k}{n}}{1 + \frac{n+1}{2n}k}.$$

Example 46.2
A \$100 par value 8-year bond with 6% semiannual coupons is selling for \$97. Find the yield rate convertible semiannually.

Solution.
We have $g = 3\%,\ n = 16,\ k = \frac{P-C}{C} = \frac{97-100}{100} = -0.03$. An initial starting guess for the Newton-Raphson method is

$$i_0 = \frac{0.03 - (-0.03/16)}{1 + \frac{16+1}{32}} = 0.0323912$$

and the iterations are given by the formulas

$$i_{s+1} = i_s\left(1 + \frac{0.03a_{\overline{16}|i_s} + (1+i_s)^{-16} - 0.97}{0.03a_{\overline{16}|i_s} + 16(i_s - 0.03)(1+i_s)^{-17}}\right).$$

We obtain the following values

s	i_s	$2i_s$
0	0.0323912	0.0647824
1	0.0324329	0.0648658
2	0.0324329	0.0648658

Thus, the yield rate convertible semiannually is 6.48658%. It is worth noting that the selling price is

$$P = 100\left(\frac{0.06}{2}\right)a_{\overline{16}|3.24329\%} + 100(1.0324329)^{-16} = \$97 \blacksquare$$

One can also detetermine the yield rate when the purchasing date is between two coupon payment dates. This situation is more complex due to the fractional period prior to the next coupon payment date. In practice, this case involves the use of the semi-theoretical method of Section 45 which leads to an implicit equation in i that can be solved by either linear interpolation or by iterations. We illustrate this in the next example.

Example 46.3
A \$100 par value 10-year bond with 8% semiannual coupons was issued on March 1, 2001. The market price on May 15, 2003 is \$88. Find the yield rate if the bond in bought on that date.

Solution.
The price of the bond on March 1, 2003 immediately after a coupon payment is

$$100 + (4 - 100i)a_{\overline{16}|i}.$$

The number of days from March 1, 2003 to May 15, 2003 is 75, while the number of days from March 1, 2003 to September 1, 2003 is 184. Using the semi-theoretical method of Section 45, we have the equation

$$[100 + (4 - 100i)a_{\overline{16}|i}](1+i)^{\frac{75}{184}} - \frac{75}{184}\cdot 4 = 88.$$

Let's solve this last equation by using linear interpolation. Let

$$f(i) = [100 + (4 - 100i)a_{\overline{16}|i}](1+i)^{\frac{75}{184}} - \frac{75}{184} \cdot 4 - 88.$$

By trial and error we find $f(0.05) = 23.43 > 0$ and $f(0.052) = -2.45 < 0$. Thus, using linear interpolation we find

$$i \approx 0.05 - 23.43 \times \frac{0.052 - 0.05}{-2.45 - 23.43} = 0.0518.$$

So the nominal yield rate convertible semiannually is 10.36%. The price on May 15,2003 is

$$100 + (4 - 100 \times 0.0518)a_{\overline{16}|0.0518}(1.0518)^{\frac{75}{184}} - \frac{75}{184} \cdot 4 = \$87.56$$

a roundoff error of 0.44 ■

We finally consider finding the yield rate of a bond with coupons being reinvested at a different interest rate. Consider the situation in which a bond is purchased for P, coupons of Fr are paid at the end of each period for n periods, the bond is redeemed for C at the end of n periods, and the coupons are reinvested at rate j. Denoting the yield rate (considering reinvestment) by i', we would have the equation of value:

$$P(1+i')^n = Frs_{\overline{n}|j} + C$$

since both sides represent the value of the investment at the end of n periods.

Example 46.4
A \$100 par value 10-year bond with 8% semiannual coupons is selling for \$90 and the coupons can be reinvested at only 6% convertible semi-annually. Find the yield rate taking in consideration reinvestment rates.

Solution.
We have $90(1+i')^{20} = 4s_{\overline{20}|0.03} + 100 \rightarrow (1+i')^{20} = \frac{4(26.8704)+100}{90} = 2.30535 \rightarrow i' = 0.04265$. Thus, the yield rate is $2(0.04265) = 0.0853 = 8.53\%$ convertible semi-annually. Thus, the yield rate of 9.48% found in Example 44.1 drops to 8.53% if the coupons can be reinvested at 6% ■

Practice Problems

Problem 46.1
A \$100 par value 12-year bond with 10% semiannual coupons is selling for \$110. Find the yield rate convertible semiannually.

Problem 46.2
Recompute the overall yield rate to an investor purchasing the bond in the previous problem, if the coupons can be reinvested at only 7% convertible semiannually.

Problem 46.3
A \$100 bond with annual coupons is redeembale at par at the end of 15 years. At a purchase price of \$92 the yield rate is exactly 1% more than the coupon rate. Find the yield rate on the bond.

Problem 46.4
An investor buys 20-year bonds, each having semiannual coupons and each maturing at par. For each bond the purchase price produces the same yield rate. One bond has a par value of \$500 and a coupon of \$45. The other bond has a par value of \$1000 and a coupon of \$30. The dollar amount of premium on the first bond is twice as great as the dollar amount of discount on the second bond. Find the yield rate convertible semiannually.

Problem 46.5
A 20 year bond with a 20,000 par value pays semi-annual coupons of 500 and is redeemable at par. Audrey purchases the bond for 21,000. Calculate Audrey's semi-annual yield to maturity on the bond. Use Salesman's formula.

Problem 46.6
A 10 year bond with a par value of 1000 is redeemable at par and pays annual coupons of 65. If Jamie purchased the bond for 950 and she reinvests the coupons at 4% annual effective rate, calculate the actual yield that Jamie will receive taking into account the reinvestment rate.

Problem 46.7 ‡
A 1000 par value 10-year bond with coupons at 5%, convertible semiannually, is selling for 1081.78. Calculate the yield rate convertible semiannually.

Problem 46.8 ‡
Dan purchases a 1000 par value 10-year bond with 9% semiannual coupons for 925. He is able to reinvest his coupon payments at a nominal rate of 7% convertible semiannually.
Calculate his nominal annual yield rate convertible semiannually over the ten-year period.

Problem 46.9
Suppose a 10000 par bond has 6% semi-annual coupons and matures in 10 years. What is the yield rate if the price is 9000? Compute both exactly and using the Salesman's method.

Problem 46.10
A \$1000 par value 10-year bond with 7% semiannual coupons was issued on October 1, 2006. The market price on April 28, 2006 is \$1020. Find the yield rate if the bond in bought on that date.

Problem 46.11
John buys a \$1000 bond that pays coupons at a nominal interest rate of 7% compounded semiannually. The bond is redeemable at par in 15 years. The price he pays will give him a yield of 8% compounded semiannually if held to maturity. After 5 years, he sells the bond to Peter, who desires a yield of 6% on his investment.
(a) What price did John pay?
(b) What price did Peter pay?
(c) What yield did John realize?

47 Callable Bonds and Serial Bonds

Callable or **redeemable** bonds are bonds that can be redeemed or paid off by the issuer prior to the bonds' maturity date. The earliest such **call date** will generally be several years after the issue date. Call dates are usually specified by the bond issuer. When an issuer calls its bonds, it pays investors the call price (usually the face value of the bonds) together with accrued interest to date and, at that point, stops making interest payments.
An issuer may choose to redeem a callable bond when current interest rates drop below the interest rate on the bond. That way the issuer can save money by paying off the bond and issuing another bond at a lower interest rate. This is similar to refinancing the mortgage on your house so you can make lower monthly payments.
Callable bonds are more risky for investors than non-callable bonds because an investor whose bond has been called is often faced with reinvesting the money at a lower, less attractive rate. As a result, callable bonds often have a higher annual return to compensate for the risk that the bonds might be called early.
Callable bonds present a problem when calculating prices and yields, because the call date of the bond is unknown. Since the issuer has an option whether or not to call the bond, the investor should assume that the issuer will exercise that option to the disadvantage of the investor, and should calculate the yield rate and/or price accordingly.
If the redemption value is the same at any call date, including the maturity date, then the following general principle will hold:
(i) the call date will most likely be at the earliest date possible if the bond was sold at a premium, which occurs when the yield rate is smaller than the coupon rate (issuer would like to stop repaying the premium via the coupon payments as soon as possible).
(ii) the call date will most likely be at the latest date possible if the bond was sold at a discount, which iccurs when the yield rate is larger than the coupon rate (issuer is in no rush to pay out the redemption value).

Example 47.1
A 15 year 1000 par bond has 7% semiannual coupons and is callable at par after 10 years. What is the price of the bond to yield 5% for the investor?

Solution.
Since the yield rate is smaller than the coupan rate, the bond is selling at premium so the earliest redemption date is the most favorable for the issuer and the least favorable for the investor. The price is the smallest of $1000 + (35 - 25)a_{\overline{n}|0.025}$ for $20 \leq n \leq 30$, which clearly occurs when $n = 20$. Thus, the price is

$$1000 + (35 - 25)a_{\overline{20}|0.025} = 1,155.89 \blacksquare$$

If the redemption values on all the redemption dates are not equal, this principle is a little harder to apply. In this case one needs to examine all call dates. The most unfavorable date will not necessarily be either the earliest or the latest possible date. The most unfavorable call date (to the issuer) is the one that produces the smallest purchase price at the investors yield rate as illustrated in the following example.

Example 47.2
A 100 par value 4% bond with semi-annual coupons is callable at the following times:
109.00, 5 to 9 years after issue
104.50, 10 to 14 years after issue
100.00, 15 years after issue (maturity date).
What price should an investor pay for the callable bond if they wish to realize a yield rate of
(1) 5% payable semi-annually and
(2) 3% payable semi-annually?

Solution.
(1) Since the market rate is better than the coupon rate, the bond would have to be sold at a discount and as a result, the issuer will wait until the last possible date to redeem the bond:

$$P = 2.00a_{\overline{30}|2.5\%} + 100.00v_{0.025}^{30} = \$89.53.$$

(2) Since the coupon rate is better than the market rate, the bond would sell at a premium and as a result, the issuer will redeem at the earliest possible date for each of the three different redemption values:

$$\begin{aligned} P =& 2.00a_{\overline{10}|1.5\%} + 109.00v_{0.015}^{10} = \$112.37 \\ P =& 2.00a_{\overline{20}|1.5\%} + 104.50v_{0.015}^{20} = \$111.93 \\ P =& 2.00a_{\overline{30}|1.5\%} + 100.00v_{0.015}^{30} = \$112.01 \end{aligned}$$

In this case, the investor would only be willing to pay \$111.93 ■

Serial Bonds
Serial bonds are bonds issued at the same time but with different maturity dates. The price of any individual bond is found by the methods already discussed earlier in this chapter provided that the redemption date is known. The price of the entire serial bonds is just the sum of the values of the individual bonds.
We next consider finding a formula for pricing an issue of serial bonds with m different redemption dates. Let P_t be the purchase price of the tth bond; C_t be the redemption value and K_t the present value of C_t. Then by Makeham's formula the price of the tth bond is

$$P_t = K_t + \frac{g}{i}(C_t - K_t).$$

Summing from $t = 1$ to $t = m$ to obtain the price of the entire issue of a serial bond

$$\sum_{t=1}^{m} P_t = \sum_{t=1}^{m} K_t + \frac{g}{i}\left(\sum_{t=1}^{m} C_t - \sum_{t=1}^{m} K_t\right)$$

or

$$P' = K' + \frac{g}{i}(C' - K')$$

where

$$P' = \sum_{t=1}^{m} P_t,\ K' = \sum_{t=1}^{m} K_t, \text{ and } C' = \sum_{t=1}^{m} C_t.$$

Example 47.3
A \$10,000 serial bond is to be redeemed in \$1000 installments of principal per half-year over the next five years. Interest at the annual rate of 12% is paid semi-annually on the balance outstanding. How much should an investor pay for this bond in order to produce a yield rate of 8% convertible semi-annually?

Solution.
We have

$$\begin{aligned}P' =& \sum_{t=1}^{10} 1000v_{0.04}^t + \frac{0.06}{0.04}\left[10,000 - \sum_{t=1}^{10} 1000v_{0.04}^t\right]\\ =& 1000a_{\overline{10}|0.04} + 1.5[10,000 - 1000a_{\overline{10}|0.04}]\\ =& 15,000 - 500a_{\overline{10}|0.04} = 10,944.58 \blacksquare\end{aligned}$$

Example 47.4
A 5000 serial bond with 10% annual coupons will be redeemed in five equal installments of 1100 beginning at the end of the 11th year and continuing through the 15th year. The bond was bought at a price P to yield 9% annual effective. Determine P.

Solution.
We have: $F = 1000, C = 1100, i = 0.09$ and $r = 0.10$. Thus, $g = \frac{1000(0.10)}{1100} = \frac{1}{11}$. Hence, $C' = 5500$

$$\begin{aligned}K' =& \sum_{t=11}^{15} 1100v_{0.09}^t\\ =& 1100(a_{\overline{15}|0.09} - a_{\overline{10}|0.09}) = 1807.33\end{aligned}$$

The price is

$$P' = 1807.33 + \frac{1}{0.99}(5500 - 1807.33) = 5537.30 \blacksquare$$

Practice Problems

Problem 47.1
A \$1000 par value bond has 8% semiannual coupons and is callable at the end of the 10th through the 15th years at par.
(a) Find the price to yield 6% convertible semiannually.
(b) Find the price to yield 10% convertible semiannually.

Problem 47.2
A \$1000 par value 8% bond with quarterly coupons is callable five years after issue. The bond matures for \$1000 at the end of ten years and is sold to yield a nominal rate of 6% convertible quarterly under the assumption that the bond will not be called. Find the redemption value at the end of five years that will provide the purchaser the same yield rate.

Problem 47.3
A \$1000 par value 4% bond with semiannual coupons matures at the end of 10 years. The bond is callable at \$1050 at the ends of years 4 through 6, at \$1025 at the ends of years 7 through 9, and at \$1000 at the end of year 10. Find the maximum price that an investor can pay and still be certain of a yield rate of 5% convertible semiannually.

Problem 47.4
A \$1000 par value 6% bond with semiannual coupons is callable at par five years after issue. It is sold to yield 7% semi-annual under the assumption that the bond will be called. The bond is not called and it matures at the end of 10 years. The bond issuer redeems the bond for $1000 + X$ without altering the buyer's yield rate of 7% convertible semiannually. Find X.

Problem 47.5
A 15 year bond can be called at the end of any year from 10 through 14. The bond has a par value of 1000 and an annual coupon rate of 5%. The bond is redeemable at par and can also be called at par.
(a) Calculate the price an investor would pay to yield 6%.
(b) Calculate the price an investor would pay to yield 4%.

Problem 47.6
A 15 year bond can be called at the end of any year from 12 through 14. The bond has a par value of 1000 and an annual coupon rate of 5%. The bond is redeemable at par and can be called at 1100.
(a) Calculate the price an investor would pay to yield 6%.
(b) Calculate the price an investor would pay to yield 4%.

Problem 47.7
A callable 20 year bond which matures for 1000 pays annual coupons of 80. The bond is callable in years 10 through 12 at 1100 and in years 13 through 15 at 1050. The bond is bought to yield 4% annually.
Calculate the purchase price of the bond.

Problem 47.8 ‡
Matt purchased a 20-year par value bond with semiannual coupons at a nominal annual rate of 8% convertible semiannually at a price of 1722.25. The bond can be called at par value X on any coupon date starting at the end of year 15 after the coupon is paid. The price guarantees that Matt will receive a nominal annual rate of interest convertible semiannually of at least 6%. Calculate X.

Problem 47.9 ‡
Toby purchased a 20-year par value bond with semiannual coupons at a nominal annual rate of 8% convertible semiannually at a price of 1722.25. The bond can be called at par value 1100 on any coupon date starting at the end of year 15.
What is the minimum yield that Toby could receive, expressed as a nominal annual rate of interest convertible semiannually?

Problem 47.10 ‡
Sue purchased a 10-year par value bond with semiannual coupons at a nominal annual rate of 4% convertible semiannually at a price of 1021.50. The bond can be called at par value X on any coupon date starting at the end of year 5. The price guarantees that Sue will receive a nominal annual rate of interest convertible semiannually of at least 6%.
Calculate X.

Problem 47.11 ‡
Mary purchased a 10-year par value bond with semiannual coupons at a nominal annual rate of 4% convertible semiannually at a price of 1021.50. The bond can be called at par value 1100 on any coupon date starting at the end of year 5.
What is the minimum yield that Mary could receive, expressed as a nominal annual rate of interest convertible semiannually?

Problem 47.12 ‡
A 1000 par value bond with coupons at 9% payable semiannually was called for 1100 prior to maturity.
The bond was bought for 918 immediately after a coupon payment and was held to call. The nominal yield rate convertible semiannually was 10%.
Calculate the number of years the bond was held.

Problem 47.13 ‡
A 1000 par value bond pays annual coupons of 80. The bond is redeemable at par in 30 years, but is callable any time from the end of the 10th year at 1050.
Based on her desired yield rate, an investor calculates the following potential purchase prices, P :
Assuming the bond is called at the end of the 10th year, $P = 957$
Assuming the bond is held until maturity, $P = 897$
The investor buys the bond at the highest price that guarantees she will receive at least her desired yield rate regardless of when the bond is called.
The investor holds the bond for 20 years, after which time the bond is called.
Calculate the annual yield rate the investor earns.

Problem 47.14
Which of the following are true:

(I) If a callable bond is called or redeemed at any date other than that used to calculate the purchase price, the buyer's yield will be higher than that used to calculate the yield rate.
(II) If the redemption value of a callable bond is equal at all possible redemption dates and the bond sells at a premium, the price of the bond is calculated by assuming the redemption date is the last possible date.
(III) To calculate the price of a callable bond, a buyer should calculate the price at each and every possible redemption date and take the lowest price.

Problem 47.15
A 5000 par value 18-year bond has 7% semiannual coupons, and is callable at the end of the 12th through the 17th years at par.
(a) Find the price to yield 6% convertible semiannually.
(b) Find the price to yield 8% convertible semiannually.
(c) Find the price to yield 8% convertible semiannually, if the bond pays a pre- mium of 250 if it is called.
(d) Find the price to yield 6% convertible semiannually, if the bond pays a premium of 250 if it is called at the end of years 12 through 14, and a premium of 150 if it is called at the end of years 15 or 16. (It may still be called at the end of year 17 without premium, and otherwise will mature at the end of year 18.)

Problem 47.16
Find the price of a $1000 issue of 5.25% bonds with annual coupons which will be redeemed in 10 annual installments at the end of the 11th through the 20th years from the issue date at 105. The bonds are bought to yield 7% effective.

Problem 47.17
ABC is issuing a \$1000 bond with a maturity of 25 year and annual 10% coupon. The bond is callable with the first year call price equal to the offering price plus the coupon; thereafter the call price decreases by equal amounts to equal par at year 20; thereafter the call price is equal to par. If the bond were sold at \$950 what would be the call prices for each year?

Stocks and Money Market Instruments

One of the reasons investors are interested in trading with stocks because of its high rate of return (about 10%). Stocks are examples of financial instruments or securities, terms which we define next.
A **financial instrument** is a monetary claim that one party has on another. It is a financial asset for the person who buys or holds it, and it is a financial liability for the company or institution that issues it. For example, a share of Microsoft stock that you own. The stock gives you a share of Microsoft's assets and the right to receive a share of dividends (profits), if Microsoft is paying any. To Microsoft owners, a stock is an obligation to include you in their divident payments.
A **security** is a tradeable financial instrument, like a bond or a share of stock.
A **financial market** is a market in which financial assets are traded. In addition to enabling exchange of previously issued financial assets, financial markets facilitate borrowing and lending by facilitating the sale by newly issued financial assets. Examples of financial markets include the New York Stock Exchange (resale of previously issued stock shares), the U.S. government bond market (resale of previously issued bonds), and the U.S. Treasury bills auction (sales of newly issued T−bills). A **financial institution** is an institution whose primary source of profits is through financial asset transactions. Examples of such financial institutions include discount brokers (e.g., Charles Schwab and Associates), banks, insurance companies, and complex multi−function financial institutions such as Merrill Lynch.
Stocks are financial instruments that make the holder a co−owner of the company that issued them. They entitle the holder to a claim on the assets (and implicitly the future profits) of the company. Stocks do not involve the repayment of a debt or the payment of interest. (Some stocks pay dividends, which are shares of the company's profits, but people typically hold stocks not for the dividends but for the hope of reselling them later at a higher price.)
In this chapter, we introduce two types of stocks: preferred and common stocks. We also discuss trading stocks. More specifically, we will discuss buying stocks by borrowing money (buying on margin) and selling stocks that we don't own (short sales of socks). We conclude this chapter by introducing some of the financial instrument used in today's financial markets.

48 Preferred and Common Stocks

Three common types of securities which have evolved in the financial markets: bonds, preferred stocks, and common stocks. Bonds have been the topic of the previous chapter. In this section, we discuss preferred stocks and common stocks.

Preferred Stock

This is a type of security which provides a fixed rate of return (similar to bonds). However, unlike bonds, it is an ownership security (bonds are debt security). Generally, preferred stock has no maturity date, although on occasion preferred stock with a redemption provision is issued. The periodic payment is called a **dividend,** because it's being paid to an owner.

Preferred stock is typically a higher ranking stock than common stock. Preferred stock may or may not carry voting rights, and may have superior voting rights to common stock. Preferred stock may carry a dividend that is paid out prior to any dividends to common stock holders. Preferred stock may have a convertibility feature into common stock. Preferred stock holders will be paid out in assets before common stockholders and after debt holders in bankruptcy.

In general, there are four different types of preferred stock:

• **Cumulative preferred stock.** Preferred stock on which dividends accrue in the event that the issuer does not make timely dividend payments. Most preferred stock is cumulative preferred.

• **Non-cumulative preferred stock.** Preferred stock for which unpaid dividends do not accrue.

• **Participating preferred stock.** This is a kind of preferred stock that pays a regular dividend and an additional dividend when the dividend on the common stock exceeds a specific amount.

• **Convertible preferred stock.** Convertible preferred stock has a privilege similar to convertible bonds. Owners have the option to convert their preferred stock to common stock under certain conditions.

Since a preferred stock has no redemption date, it pays dividend forever. Their cash flows are, therefore, those of a perpetuity. Thus, the price of a preferred stock is the present value of future dividends of a perpetuity which is given by the formula

$$P = \frac{Fr}{i}.$$

Example 48.1

A preferred stock pays a coupon of 50 every six months. Calculate the price of the preferred stock using a yield rate of 5% convertible semi-annually.

Solution.

The price of the stock is

$$P = \frac{Fr}{i} = \frac{50}{0.025} = 2000 \blacksquare$$

Common stock
Common stock is another type of ownership security. It does not earn a fixed dividend rate as preferred stock does. Dividends on this type of stock are paid only after interest payments on all bonds and other debt and dividends on preferred stock are paid. The dividend rate is flexible, and is set by the board of directors at its discretion.
Value of a share of common stock is being based on the **dividend discount model.** This model values shares at the discounted value of future dividend payments. That is, the value of a share is the present value of all future dividends.
Consider a situation where a dividend D is paid at the end of the current period. Assume that the next dividend payments follow a geometric progression with common ratio $1+k$ indefinitely and the stock is purchased at a yied rate of i with $-1 < k < i$. Then the theoretical price of the stock is

$$P = D \lim_{n\to\infty} \frac{1-\left(\frac{1+k}{1+i}\right)^n}{i-k} = \frac{D}{i-k}.$$

We point out that the most common frequency of dividend payments on both preferred and commom stock in the US is quarterly.

Example 48.2
A common stock is currently earning \$1 per share and the dividend is assumed to increase by 5% each year forever. Find the theoretical price to earn an investor an annual effective yield rate of 10%.

Solution.
The theoretical price is

$$P = D\frac{1}{i-k} = \frac{1}{0.10-0.05} = 20 \blacksquare$$

It is unrealistic to project constant percentage increases in dividends indefinitely into the future. As corporations increase in size and become more mature, the rate of growth will generally slow down.

Example 48.3
A common stock pays annual dividends at the end of each year. The earnings per share for the most recent year were 8 and are assumed to grow at a rate of 10% per year, forever. The dividend will be 0% of earnings for each of the next 10 years, and 50% of earnings thereafter. What is the theoretical price of the stock to yield 12%?

Solution.
The theoretical price of the stock which is the present value of all future dividends is

$$\begin{aligned}\sum_{k=11}^{\infty} 4(1.1)^k(1.12)^{-k} =& \sum_{k=11}^{\infty} 4\left(\frac{1.1}{1.12}\right)^k \\ =&4\left(\frac{1.1}{1.12}\right)^{11}\left[\frac{1}{1-\left(\frac{1.1}{1.12}\right)}\right] = 183.73 \blacksquare\end{aligned}$$

Example 48.4
A common stock is currently earning \$4 per share and will pay \$2 per share in dividends at the end of the current year. Assuming that the earnings of the corporation increase at the rate of 5% for the first five years, 2.5% for the second five years and 0% thereafter. Assume that the corporation plans to continue to pay 50% of its earnings as dividends, find the theoretical price to earn an investor an annual effective yield rate of 10%.

Solution.
The theoretical price is

$$\begin{aligned}&2\left[\frac{1-\left(\frac{1.05}{1.10}\right)^5}{0.10-0.05}\right]+2\frac{(1.05)^5}{(1.10)^5}\left[\frac{1-\left(\frac{1.025}{1.10}\right)^5}{0.10-0.025}\right] \\ &+\frac{2(1.05)^5(1.025)^5}{(1.10)^{10}}\cdot\frac{1}{0.10} = \$25.72 \blacksquare\end{aligned}$$

Practice Problems

Problem 48.1
A preferred stock pays a \$10 dividend at the end of the first year, with each successive annual dividend being 5% greater than the preceding one. What level annual dividend would be equivalent if $i = 12\%$?

Problem 48.2
A preferred stock will pay a dividend of 100 today. The stock also provides that future dividends will increase at the rate of inflation. The stock is purchased to yield 8% and assuming an inflation rate of 3%.
Calculate the price of the stock today (before payment of the dividend).

Problem 48.3
A common stock pays annual dividends at the end of each year. The earnings per share in the year just ended were \$6. Earnings are assumed to grow 8% per year in the future. The percentage of earnings paid out as a dividend will be 0% for the next five years and 50% thereafter. Find the theoretical price of the stock to yield an investor 15% effective.

Problem 48.4
A common stock is purchased at a price equal to 10 times current earnings. During the next 6 years the stock pays no dividends, but earnings increase 60%. At the end of 6 years the stock sold at a price equal 15 times earnings. Find the effective annual yield rate earned on this investment.

Problem 48.5
A company's common stock pays a dividend of \$2 each year. The next dividend will be paid in one year. If the dividend is expected to increase at 5% per year, calculate the value of the stock at a 10% yield rate.

Problem 48.6
A company's current earnings are \$5 per share. They expect their earnings to increase at 10% per year. Once their earnings are \$10 per share, they will payout 75% as a dividend. Calculate the value of the stock to yield 15%.

Problem 48.7
Thompson Corporation currently pays a dividend of 3 on earnings of 5 per year. Thompson's earnings are expected to increase at a rate of 6% per year with 60% of earnings paid as dividends. Calculate the theoretical price of the common stock to yield 10%.

Problem 48.8
A common stock is currently earning \$4 per share and will pay \$2 per share in dividends at the end of the current year. Assuming that the earnings of the corporation increase 5% per year indefinitely and that the corporation plans to continue to pay 50% of its earnings as dividends, find the theoretical price to earn an investor an annual effective yield rate of (1) 10%, (2) 8%, and (3) 6%

Problem 48.9
A common stock is purchased on January 1, 1992. It is expected to pay a dividend of 15 per share at the end of each year through December 31, 2001. Starting in 2002 dividends are expected to increase K% per year indefinitely, $K < 8\%$. The theoretical price to yield an annual effective rate of 8% is 200.90. Calculate K.

Problem 48.10
On January 1 of each year Company ABC declares a dividend to be paid quarterly on its common shares. Currently, 2 per share is paid at the end of each calendar quarter. Future dividends are expected to increase at the rate of 5% per year. On January 1 of this year, an investor purchased some shares at X per share, to yield 12% convertible quarterly. Calculate X.

Problem 48.11 ‡
The dividends of a common stock are expected to be 1 at the end of each of the next 5 years and 2 for each of the following 5 years. The dividends are expected to grow at a fixed rate of 2% per year thereafter.
Assume an annual effective interest rate of 6%. Calculate the price of this stock using the dividend discount model.

Problem 48.12 ‡
A commom stock sells for 75 per share assuming an annual effective interest rate of i. Annual dividends will be paid at the end of each year forever.
The first dividend is 6, with each subsequent dividend 3% greater than the previous year's dividend. Calculate i.

49 Buying Stocks

In this section, we discuss the costs associated with buying stocks. One way to buy stocks is by **buying on margin** (i.e. buying stocks with borrowed money).
When an investor is buying an asset, he is said to have a **long** position in the asset. If the investor is selling an asset he is said to have a **short** position in that asset. In the stock and bond markets, if you short an asset, it means that you borrow it, sell the asset, and then later buy it back. Short sales will be discussed in Section 50.
Now, depending upon whether an investor is buying or selling a security, he will get different price quotes. A **bid** price is the price at which a broker is ready to purchase the security. Thus, this is the selling price for an investor. The **ask** price is the price at which a broker is ready to sell the security. The ask is the buying price paid by an investor. The difference between the two prices is called the **bid-ask spread** (bid price is lower than the asked price, and the spread is the broker's profit).

Example 49.1
MBF stock has a bid price of \$10.35 and an ask price of \$11.00. Assume there is a \$5 brokerage commission.
(a) What is the bid-ask spread?
(b) What amount will you pay to buy 50 shares?
(c) What amount will you receive for selling 50 shares?
(d) Suppose you buy 50 shares, then immediately sell them with the bid and ask price being the same as above. What is your round-trip transaction cost?

Solution.
(a) The bid-ask spread is $11.00 - 10.35 = \$0.65$.
(b) You pay $50 \times 11.00 + 5 = \$555.00$
(c) You receive $50 \times 10.35 - 5 = \$512.50$.
(d) For each stock, you buy at \$11.00 and sell it an instant later for \$10.35. The total loss due to the ask-bid spread: $50(11.00 - 10.35) = \$32.50$. In addition, you pay \$5 twice. Your total transaction cost is then $32.50 + 2(5) = \$42.50$ ■

Buying on Margin
When buying on margin, the investor borrows part of the purchase price of the stock from a broker or a brokerage firm and contributes the remaining portion (the **initial margin requirement** or simply the **margin**) which is kept in an account with the brokerage firm. If you buy Q shares at the price of P (per share) with **initial margin percentage** of $IM\%$, then your initial margin requirement IM is given by the formula

$$IM = Q \cdot P \cdot (IM\%).$$

Example 49.2
Suppose you want to buy 100 shares on margin at the price of \$50 per share. Assuming that the initial margin percentage is 50%, what was your initial margin requirement?

Solution.
You initial margin requirement IM is $IM = 100(50)(0.5) = \$2500$ ■

The brokerage firm is making money on your loan in terms of interest. Moreover, your stock will be hold as the collateral against the loan. If you default, the firm will take the stock.
There are risks associated to buying stock on margin. The price of your stock could always go down. By law, the brokerage firm will not be allowed to let the value of the collateral (the price of your stock) go down below a certain percentage of the loan value. This percentage is called the **maintenance margin**. If the stock drops below that set amount, the brokerage firm will issue a margin call on your stock.
The margin call means that you will have to pay the brokerage firm the amount of money necessary to bring the brokerage firms risk down to the allowed level. If you don't have the money, your stock will be sold to pay off the loan. If there is any money left, it will be sent to you. In most cases, there is little of your original investment remaining after the stock is sold.
We define the **actual margin** AM by the formula

$$AM = \frac{\text{Market Value of Assets} - \text{Loan}}{\text{Market Value of Assets}}$$

If $AM < MM$, then the investor is undermargined (he or she receives a margin call from his or her broker) and must take corrective action (pay off part of loan or deposit more collateral). If the investor fails to respond, then the broker will close the investor's account. If $AM > IM$, then the investor is overmargined and can withdraw cash or buy more shares.

Example 49.3
John wants to purchase 100 stocks of IBM. The current price of a stock is \$100. He has \$6000 on hand, so he decides to borrow \$4000. The maintenance marginal is set to be 50%.
(a) What is John's liability and equity?
(b) Suppose the stock falls to \$70. What is the actual margin? Would a margin call be in order?
(c) How far can the stock price fall before a margin call?

Solution.
(a) John's liability is \$4000 and his equity is \$6000. So John is contributing $\frac{6000}{10000} = 60\%$ (the margin) of the purchasing price.
(b) If the stock falls to \$70, then John's liability is still \$4000 but his equity is reduced to \$3000 so that the new margin is $\frac{3000}{7000} = 43\% < 50\%$. Since this is less than the maintenance margin, the

broker will issue a margin call.
(c) Let P be the original price. Then the margin in this case is

$$\frac{100P - 4000}{100P}.$$

Setting this to 0.5 and solving for P we find $P = \$80$. Thus, anytime the stock drops below \$80, John will receive a margin call ■

Example 49.4
John wants to purchase 100 stocks of IBM. The current price of a stock is \$100. He has \$6000 on hand, so he decides to borrow \$4000. The broker charges an annual effective rate of interest of 10%. Suppose the stock goes up by 30% in one year period.
(a) What is the end-year-value of the shares?
(b) Find the total of loan repayment and interest.
(c) What is the rate of return?
(d) What is the rate of return if not buying on margin?

Solution.
(a) The end-year-value of the shares is $100[100 + (30\%)(100)] = \$13,000$.
(b) The total of loan repayment and interest is $4000 + 4000(10\%) = \$4,400$.
(c) The rate of return is

$$\frac{13000 - 10400}{6000} = 43.3\%$$

(d) If not buying on margin, the rate of return is

$$\frac{13000 - 10000}{10000} = 30\% \blacksquare$$

The reason why an investor buys on margin is his wish to invest more than what the investor's money would allow. Buying on margin could mean a huge return when stock prices go up. But there is the risk that you could lose your original investment when stock price goes down. As with any stock purchase there are risks, but when you are using borrowed money, the risk is increased.

Practice Problems

Problem 49.1
MBF stock has a bid price of \$10.35 and an ask price of \$11.00. Assume there is a 0.1% brokerage commission on the bid or ask price.
(a) What amount will you pay to buy 50 shares?
(b) What amount will you receive for selling 50 shares?
(c) Suppose you buy 50 shares, then immediately sell them with the bid and ask price being the same as above. What is your round-trip transaction cost?

Problem 49.2
MBF stock has a bid price of \$10.35 and an ask price of \$11.00. At what price can the market-maker purchase a stock? At what price can a market-maker sell a stock? What is the market-maker profit from buying and immediately selling 50 shares?

Problem 49.3
Suppose the spot ask exchange rate is \$2.10 = £1.00 and the spot bid exchange rate is \$2.07 = £1.00. If you were to buy \$5,000,000 worth of British pounds and then sell them five minutes later without the bid or ask changing, how much of your \$5,000,000 would be "eaten" by the bid-ask spread?

Problem 49.4
Consider the transaction of Example 49.4 but now assume that there is no change in the stock price in one year period.
(a) What is the end-year-value of the shares?
(b) Find the total of loan repayment and interest.
(c) What is the rate of return?
(d) What is the rate of return if not buying on margin?

Problem 49.5
Consider the transaction of Example 49.4 but now assume that there is a decrease of 30% in the stock price in one year period.
(a) What is the end-year-value of the shares?
(b) Find the total of loan repayment and interest.
(c) What is the rate of return?
(d) What is the rate of return if not buying on margin?

Problem 49.6
On July 24, 2002, you bought 100 shares of Microsoft on margin at the price of \$45 per share. Assuming that the initial margin percentage is 50%, what was your initial margin requirement?

Problem 49.7
Using Problem 49.6, suppose that On July 25, 2002, Microsoft closed at \$42.83. What was your actual margin on that day?

Problem 49.8
Using Problem 49.6, suppose that the maintenance margin for Microsoft is 40%. Did you receive a margin call on July 25, 2002?

Problem 49.9
On December 30, 1998, you decided to bet on the January effect, a well−known empirical regularity in the stock market. On that day, you bought 400 shares of Microsoft on margin at the price of \$139 per share. The initial margin requirement is 55% and the maintenance margin is 30%. The annual cost of the margin loan is 6.5%.
(a) Determine your initial margin requirement.
(b) To what price must Microsoft fall for you to receive a margin call?
(c) On January 8, 1999, Microsoft climbed to \$151.25. What was the actual margin in your account on that day?
(d) On January 9, 1999, you sold your Microsoft shares at the price of \$150.50 per share. What was the return on your investment?
(e) Recalculate your answer to part (d) assuming that you made the Microsoft stock purchase for cash instead of on margin.

50 Short Sales

In this section we discuss trading security over short period of time when the security is likely to decline in value.
A **short sale** is generally a sale of a security by an investor who does not actually own the security. To deliver the security to the purchaser, the short seller will borrow the security. The short seller later "pays off" the loan by returning the security to the lender, typically by purchasing the sold security back on the open market (hopefully at a lower price). This process of buying back the security is called **covering the short** because the seller is no longer at risk to movements in the security.
One problem with short sales is determining the yield rate. If the transaction occurs exactly as stated, then technically the yield rate does not exist. We illustrate this in the following example.

Example 50.1
An investor sells a stock short for \$1000 and buys it back for \$800 at the end of one year. What is the yield rate?

Solution.
A possible equation of value is $1000(1+i) = 800$ which yields $i = -20\%$ which is clearly unreasonable since the seller made \$200 in profit from the transaction.
Now, if we reverse the transaction and solve the equation of value $800(1+i) = 1000$ we obtain $i = 25\%$. But this equation states that there was \$200 profit on an \$800 investment, but there never was an \$800 investment ■

Short sales rarely occur as illustrated above. In practice, the short seller is required to make a deposit (i.e. a collateral) of a percentage of the price at the time the short sale is made. This deposit is called the **margin**, and cannot be accessed by the short seller until the short sale is covered. However, the short seller will be credited with interest on the margin deposit which will increase the yield rate.
Margins are usually expressed as a percentage of the value of the security. For example, governmental regulations in the United States require the short seller to make a deposit of 50% of the price at the time the short sale is made. In the example above, the short seller will have to deposit a margin of \$500 at the time the short sale.
The short seller is not allowed to earn interest on the proceeds from the short sale. Governmental regulations require that these proceeds remain in a non-interest bearing account until the short position is covered, at which time these funds will be used for the purchase necessary to cover the short. Any leftover money is profit on the transaction, while a negative balance would be loss on the transaction.

If coupons or dividends are paid on the security while you are short it, you must pay them to the party you borrowed the security from. Dividend payments are often deducted from the margin account.
In order to find the yield rate on a short sale we introduce the following notation:
- M = Margin deposit when sell was made ($t = 0$)
- S_0 = Proceeds from short sale
- S_t = Cost to repurchase stock at time t
- d_t = Dividend at time t
- i = Periodic interest rate of margin account
- j = Periodic yield rate of short sale

If a short seller repurchases stock after n periods, then the equation of value at time $t = n$ is:

$$M(1+j)^n = M(1+i)^n + S_0 - S_n - \sum_{t=1}^{n} d_t(1+i)^{n-t}.$$

Example 50.2 ‡
Bill and Jane each sell a different stock short for a price of 1000. For both investors, the margin requirement is 50%, and interest on the margin is credited at an annual effective rate of 6%. Bill buys back his stock one year later at a price of P. At the end of the year, the stock paid a dividend of X. Jane also buys back her stock after one year, at a price of $(P - 25)$. At the end of the year her stock paid a dividend of $2X$. Both investors earned an annual effective yield of 21% on their short sales. Calculate P and X.

Solution.
The information for Bill gives $500(1.21) = 500(1.06) + 1000 - P - X$ and for Jane gives $500(1.21) = 500(1.06) + 1000 - P + 25 - 2X$. Solving this system of two linear equations in two unknowns we find $P = 900$ and $X = 25$ ■

Example 50.3
Suppose Eddie sells 10 shares of a company short; the stock price is \$87 per share, and margin requirement is 40%. The company pays semiannual dividends of 2 per share. Eddie's margin account earns 3% nominal annual interest, convertible semiannually, and he buys the stock back after 18 months at a share price of 80. What is his annual effective rate of return?

Solution.
We have $M = 10(87)(0.4) = 348$. Thus,

$$348(1+j)^3 = 348(1.015)^3 + 870 - 800 - 20s_{\overline{3}|0.015}$$

Solving for j we find $j = 2.34\%$. Hence, the annual effective rate of return is $(1.0234)^2 - 1 = 4.73\%$ ■

Example 50.4
On 01/01/2005, Bryan sells a stock short for a price of 800. The margin requirement is 50%, and interest on the margin is credited at an annual effective interest rate of 2%. The stock pays a dividend, D, on 12/31/2005. On 01/01/2006, Bryan buys the stock back at a price of 820 to cover the short. Bryan's yield on the short sell is -10%. Calculate D.

Solution.
We are given the following information: $M = 400, S_0 = 800, S_1 = 820, i = 0.02, j = -0.10$. The equation of value is

$$400(0.90) = 400(1.02) + 800 - 820 - D.$$

Solving this equation for D we find $D = 28$ ■

Short selling by itself is a speculative endeavor and should not be entered into lightly. The short seller is counting on a significant decline in the value of the security in order to come out ahead and such an act is istself a risky one. However, investment strategies have been developed that involve combinations of long and short positions in related securities, which have greatly reduced risk and also have a good prospect to earn a reasonable return on investment. A generic name for these transactions is **hedging**. Situations of this type in which a profit is certain are called **arbitrage**. These two concepts will appear later in the book when discussing derivatives markets.
Short selling has been blamed (erroneously) for many historic market crashes, and has frequently been harshly regulated or banned.

Practice Problems

Problem 50.1
John sells a stock short for 200. A year later he purchases the stock for 160. The margin requirements are 50% with the margin account earning 4%. The stock that John sold paid a dividend of 6 during the year. Calculate John's return on the stock investment.

Problem 50.2
Don and Rob each sell a different stock short. Don sells his stock short for a price of 960, and Rob sells his short for 900. Both investors buy back their securities for X at the end of one year. In addition, the required margin is 50% for both investors, and both receive 10% annual effective interest on their margin accounts. No dividends are paid on either stock. Don's yield rate on his short sale is 50% greater than Rob's yield rate on his short sale. Calculate Don's yield rate.

Problem 50.3
Brian sells a stock short for 800 and buys it back one year later at a price of 805. The required margin on the sale is 62.5% and interest is credited on the margin deposit at an annual effective rate of i. Dividends of 15 are paid during the year. Brian's yield rate over the one year period is j. If the interest credited on the margin deposit had been $1.25i$, with everything else remaining the same, Brian's yield rate over the one-year period would have been $1.5j$. Calculate i.

Problem 50.4
An investor sells short 500 shares of stock at 10 per share and covers the short position one year later when the price of the stock has declined to 7.50. The margin requirement is 50%. Interest on the margin deposit is 8% effective. Four quarterly dividends of 0.15 per share are paid. Calculate the yield rate.

Problem 50.5
Andrew sells a stock short for 800. At the end of one year, Andrew purchases the stock for 760. During the year, Andrew's stock paid a dividend of 50. Amanda sold a different stock short 1000. At the end of the year, Amanda purchases the stock she sold short for X. Amanda's stock paid a dividend of 25 during the year. The margin requirement for both Andrew and Amanda is 60% and they both earn 10% interest on the margin account. Andrew and Amanda both received the same return. Calculate X.

Problem 50.6
Thomas sells a stock short for 1000. The margin rate is 60%. The stock does not pay dividends. After one year, Thomas purchases the stock for 880 which results in a yield of 25%.
Calculate the interest rate earned on the margin account.

Problem 50.7
Jenna sells short a stock for 1000. One year later she buys the stock for 1000. The stock pays a dividend of 10. Jenna earns 10% on her margin requirement of 60%.
Jordan sells short a stock for 500. Jordan's stock does not pay a dividend. Jordan earns 5% on his margin requirement of 60%. When Jordan buys the stock one year later, he earns a yield that is twice that earned by Jenna.
Determine the price that Jordan paid to purchase his stock.

Problem 50.8
James sells a stock short for 1000. The margin requirement is 60% and James earns 10% on the margin account. One year later he buys the stock for 790.
Jack sells a stock short for 600. The margin requirement is 50% and Jack earns 10% on the margin account. One year later she sells the stock for 450.
Both stocks pay the same dividend.
James and Jack both earn the same yield.
Determine the dividend paid on each stock.

Problem 50.9
Rachel sells a stock short for 2000. The margin requirement is 50%. During the year, Rachel's stock pays a dividend of 25. Rachel purchases the stock for 2050.
Danielle sells a stock short for 1500. Her margin requirement is 60%. During the year, Danielle's stock pays a dividend of 10. Danielle purchases the stock one year later for 1535.
Danielle earns a return on her transaction that is twice the return that Rachel earns on her transaction. Both Rachel and Danielle are paid the same interest rate on the margin account.
Calculate the interest rate earned on the margin account.

Problem 50.10
Adam has a yield on a short sale of the stock of 30%. The interest rate on the margin account is 10%. If the margin requirement had been doubled with no other changes to the transaction, determine the yield on Adam's transaction.

Problem 50.11
On the short sale of a stock, James realizes a yield of 22%. The margin requirement was 50% and he earned 10% on the margin. If the margin had been increased to 60%, but the interest earned on the margin is still 10%, calculate James' yield.

Problem 50.12 ‡
Theo sells a stock short with a current price of 25,000 and buys it back for X at the end of 1 year. Governmental regulations require the short seller to deposit margin of 40% at the time of the short

sale. No dividends incurred. The prevailing interest rate is an 8% annual rate, and Theo earns a 25% yield on the transaction. Calculate X.

Problem 50.13

On 01/01/2005, Company $XYZ's$ stock is trading at \$80 per share, and Betty sells short \$1000 worth of the stock. (Note: Shares of stock are bought and sold in fractional pieces; you do not have to buy or sell a whole number of shares.)
Betty is required to put up a margin of \$500, and it is agreed that she will receive interest on the margin at an annual effective interest rate of k%. Company $XYZ's$ stock pays a dividend of 1.45 per share on 12/31/2005. On 01/01/2006, Company $XYZ's$ stock is trading at \$76 per share, and Betty covers the short at this time for a yield of 9.75%. Calculate k.

Problem 50.14 ‡

Eric and Jason each sell a different stock short at the beginning of the year for a price of 800. The margin requirement for each investor is 50% and each will earn an annual effective interest rate of 8% on his margin account. Each stock pays a dividend of 16 at the end of the year. Immediately thereafter, Eric buys back his stock at a price of $(800 - 2X)$ and Jason buys back his stock at a price of $(800 + X)$. Eric's annual effective yield, j, on the short sale is twice Jasons annual effective yield. Calculate j.

Problem 50.15 ‡

Jose and Chris each sell a different stock short for the same price. For each investor, the margin requirement is 50% and interest on the margin debt is paid at an annual effective rate of 6%. Each investor buys back his stock one year later at a price of 760. Jose's stock paid a dividend of 32 at the end of the year while Chris's stock paid no dividends. During the 1-year period, Chris's return on the short sale is i, which is twice the return earned by Jose. Calculate i.

Problem 50.16 ‡

On January 1, 2004, Karen sold stock A short for 50 with a margin requirement of 80%. On December 31, 2004, the stock paid a dividend of 2, and an interest amount of 4 was credited to the margin account. On January 1, 2005, Karen covered the short sale at a price of X, earning a 20% return. Calculate X.

Problem 50.17

Kathy and Megan each sell short the same stock. Kathy is required to maintain a margin requirement of 50% while Megan is required to maintain a margin requirement of 75%. Assume no dividends paid on the stock and the purchase price is the same.
Which of the following are true:
(I) If no interest is paid on either margin account, Kathy's yield will be 150% of Megan's yield.

(II) If the same interest rate is paid on both margin accounts, Megan's yield will exceed 2/3 of Kathy's yield.
(III) If the same interest rate is paid on both margin accounts, Megan's yield could never exceed Kathy's yield.

51 Money Market Instruments

In this section we briefly introduce some of the financial instruments used in today's financial markets. These instruments can be analyzed with the principles developed in the earlier chapters of this book. The financial intruments introduced fall into the two categories: Money market instruments and financial derivatives. In this section we will cover money markets instruments. Basics of financial derivatives will be introduced and discussed at a later sections.

Corporations and government organizations are continually borrowing and lending money. One way for raising money is through money market instruments. **Money market instruments** are cash-equivalent investments that consist of short-term (usually under a year), very low-risk debt instruments. The **money market** is the market for buying and selling short-term loans and securities. The buyer of the money market instrument is the lender of money and the seller is the borrower of money. Below, we introduce few types of money market instruments that are traded in money markets.

Money Market Mutual Funds
A **money market mutual fund** is a mutual fund that attempts to keep its share price at $1. Professional money managers will take your cash and invest it in government treasury bills, savings bonds, certificates of deposit, and other safe and conservative short term commercial paper. They then turn around and pay you, the owner of the shares, your portion of the interest earned on those investments. Money market funds generally allow withdrawals on demand without penalty, making them an ideal place to keep money while considering other investment opportunities.
Investment in a money market fund is neither insured nor guaranteed by neither the FDIC or by any other government agency, and there can be no guarantee that the fund will maintain a stable $1 share price. It is possible to lose money by investing in the fund.
An example of money market fund investment is a **commercial paper** security issued by banks and large corporations. An example of a commercial paper is a promissory note. A **promissory note** is a written promise to repay a loan or debt under specific terms−usually at a stated time, through a specified series of payments, or upon demand.

Example 51.1
On January 31, Smith borrows $500 from Brown and gives Brown a promissory note. The note states that the loan will be repaid on April 30 of the same year, with interest at 12% per annum. On March 1 Brown sells the promissory note to Jones, who pays Brown a sum of money in return for the right to collect the payment from Smith on April 30. Jones pays Brown an amount such that Jones' yield (interest rate earned) from March 1 to the maturity date can be stated as an annual rate of interest of 15%.

Determine the amount that Jones paid to Brown and the yield rate Brown earned quoted on an annual basis. Assume all calculations are based on simple interest and a 365-day non-leap year.

Solution.
On April 30, the value of the note is

$$AV = 500\left[1+\left(\frac{89}{365}\right)(0.12)\right] = \$514.63.$$

The amount Jones paid to Brown on March 1 is

$$P\left[1+\left(\frac{60}{365}\right)(0.15)\right] = \$514.63$$

Solving for P we find $P = \$502.24$.
The yield rate earned by Brown satisfies the equation

$$502.24 = 500\left[1+\left(\frac{29}{365}\right)i\right]$$

Solving for i we find $i = 5.64\%$ ■

Certificates of Deposit
A certificate of deposit ("CD") is like a savings account in the sense that it is an FDIC (Federal Deposit Insurance Corporation) insured investment. However, a CD has a fixed term (often three months, six months, or one to five years). It is intended that the CD be held until maturity, at which time the money may be withdrawn together with the accrued interest.
CDs are available at banks and savings and loan institutions. CDs are issued in a range of denominations with the higher denominations (e.g. \$100,000) carrying the higher rates of interest than the lower denominations.
CDs carry a fixed yield rate and in this sense is considered more stable than a money market fund that carries a variable interest rate. However, there is the disadvantage of liquidity since penalties are imposed for early withdrawal.

Example 51.2
A two-year certificate of deposit pays an annual effective rate of 9%. The purchaser is offered two options for prepayment penalties in the event of early withdrawal:

A − a reduction in the rate of interest to 7%
B − loss of three months interest.

In order to assist the purchaser in deciding which option to select, compute the ratio of the proceeds under Option A to those under Option B if the certificate of deposit is surrendered:
(a) At the end of six months.
(b) At the end of 18 months.

Solution.
(a) Option's A proceeds: $(1.07)^{0.5} = 1.03441$. Option's B proceeds: $(1.09)^{.25} = 1.02178$. The ratio is

$$\frac{1.03441}{1.02178} = 1.0124.$$

So Option A is preferable.
(b) Option's A proceeds: $(1.07)^{1.5} = 1.10682$. Option's B proceeds: $(1.09)^{1.25} = 1.11374$. The ratio is

$$\frac{1.10682}{1.11374} = 0.9938.$$

So Option B is preferable ■

Guaranteed Investment Contracts (GIC)
GICs are investment instruments issued by insurance companies, usually to large investors such as pension funds. They guarantee principal and interest over a stated period of time (like CDs), so their market value does not fluctuate with movements in the interest rate. GICs are different from CDs in that they allow additional deposits during the first 3 months to a year. Like CDs, withdrawals are restricted. GICs pay higher rates of interest than CDs, but they are not insured by the FDIC against default.
Interest earned on the initial investment may be reinvested at the guaranteed rate, or it may be reinvested at some lower fixed rate.

Example 51.3
An investor deposits $1,000,000 at the beginning of each year for 5 years in a 5-year GIC with an insurance company that guarantees an annual rate 6% on the fund. Determine the lump sum payable by the investor at 5 years
(a) if the interest that is generated each year is reinvested at 6%;
(b) if the interest that is generated each year is reinvested at 4%;

Solution.
(a) If the guaranteed interest is reinvested at 6% then the lump sum payable by the investor at 5 years is

$$1{,}000{,}000\ddot{s}_{\overline{5}|0.06} = \$5,975,318.54.$$

(b) If the guaranteed interest is reinvested at 4% then the lump sum payable by the investor at 5 years is

$$5,000,000 + 60,000(1.04)^4 + 2 \times 60,000(1.04)^3 + 3 \times 60,000(1.04)^2$$
$$+4 \times 60,000(1.04) + 5 \times 60,000 = \$5,949,463.19 \blacksquare$$

Mutual Funds

A mutual fund is a professionally-managed form of collective investments that pools money from many investors and invests it in stocks, bonds, short-term money market instruments, and/or other securities. In a mutual fund, the fund manager trades the fund's underlying securities, realizing capital gains or losses, and collects the dividend or interest income. The investment proceeds are then passed along to the individual investors.
The value of a share of the mutual fund, known as the net asset value per share (NAV), is calculated daily based on the total value of the fund divided by the number of shares currently issued and outstanding.

Example 51.4

A mutual fund account has the balance \$100 on January 1. On July 1 (exactly in the middle of the year) the investor deposits \$250 into the account. The balance of the mutual fund at the end of the year is \$400.
(a) Calculate the exact dollar-weighted annual rate of interest for the year.
(b) Suppose that the balance of the fund immediately before the deposit on July 1 is \$120. Calculate the time-weighted annual rate of interest for the year.

Solution.

(a) $100(1+i) + 250(1+i)^{\frac{1}{2}} = 400 \rightarrow 100(1+i) + 250(1+i)^{\frac{1}{2}} - 400 = 0$. By the quadratic formula, $(1+i)^{\frac{1}{2}} = 1.108495283 \rightarrow (1+i) = 1.228761792 \rightarrow i = 0.228761792$.
(b) $(1+j) = (120/100)(400/370) = 1.297297297$ so the time-weighted rate is $j = 0.297297297$ $\blacksquare$

Mortgage Backed Securities (MBS)

Mortgage-backed securities(MBS) represent an investment in mortgage loans. An MBS investor owns an interest in a pool of mortgages, which serves as the underlying assets and source of cash flow for the security.
Most MBSs are issued by the following three financial companies:
• Government National Mortgage Association (Ginnie Mae), a U.S. government agency. Ginnie Mae, backed by the full faith and credit of the U.S. government, guarantees that investors receive timely payments.

• The Federal National Mortgage Association (Fannie Mae) and the Federal Home Loan Mortgage Corporation (Freddie Mac), U.S. government-sponsored enterprises. Fannie Mae and Freddie Mac also provide certain guarantees and, while not backed by the full faith and credit of the U.S. government, have special authority to borrow from the U.S. Treasury.
Mortgage-backed securities typically carry some of the highest yields of any government or agency security.

Example 51.5
A homeowner signs a 30 year mortgage that requires payments of \$971.27 at the end of each month. The interest rate on the mortgage is 6% compounded monthly. If the purchase price of the house is \$180,000 then what percentage down payment was required?

Solution.
The purchase price of the home is

$$971.27a_{\overline{360}|0.005} = 162,000.$$

The amount of down payment is

$$180,000 - 162,000 = 18,000.$$

So the down payment is 10% of the purchasing price ■

Practice Problems

Problem 51.1
Consider again Example 51.1. Suppose instead that Jones pays Brown an amount such that Jones' yield is 12%. Determine the amount that Jones paid.

Problem 51.2
You prurchase \$10,000 CD with an interest rate of 5% compounded annually and a term of one year. At year's end, the CD will have grown to what value?

Problem 51.3 ‡
A bank offers the following certificates of deposit:

Term in years	Nominal annual interest rate (convertible quarterly)
1	4%
3	5%
5	5.65%

The bank does not permit early withdrawal. The certificates mature at the end of the term. During the next six years the bank will continue to offer these certificates of deposit with the same terms and interest rates.
An investor initially deposits \$10,000 in the bank and withdraws both principal and interest at the end of six years.
Calculate the maximum annual effective rate of interest the investor can earn over the 6-year period.

Problem 51.4
An investor deposits \$100,000 in a 3-year GIC with an insurance company that guarantees an annual 4% interest rate on the fund. If the GIC specifies that interest is reinvested at 4% within the GIC, determine the lump sum payable to the investor after the three-year period.

Problem 51.5
An investor puts \$100 into a mutual fund in the first year and \$50 in the second year. At the end of the first year the mutual pays a dividend of \$10. The investor sells the holdings in the mutual fund at the end of the second year for \$180. Find the dollar weighted return.

Problem 51.6
You have a mutual fund. Its balance was \$10,000 on 31/12/1998. You made monthly contributions of \$100 at the start of each month and the final balance was \$12,000 at 31/12/1999. What was you approximate return during the year?

Problem 51.7
An investor purchases 1000 worth of units in a mutual fund whose units are valued at 4.00. The investment dealer takes a 9% "front-end load" from the gross payment. One year later the units have a value of 5.00 and the fund managers claim that the "fund's unit value has experienced a 25% growth in the past year." When units of the fund are sold by an investor, there is a redemption fee of 1.5% of the value of the units redeemed.
(a) If the investor sells all his units after one year, what is the effective annual rate of interest of his investment?
(b) Suppose instead that after one year the units are valued at 3.75. What is the return in this case?

Problem 51.8
Julie bought a house with a 100,000 mortgage for 30 years being repaid with payments at the end of each month at an interest rate of 8% compounded monthly. What is the outstanding balance at the end of 10 years immediately after the 120th payment?

Problem 51.9
The interest on a 30 year mortgage is 12% compounded monthly. The mortgage is being repaid by monthly payments of 700.
Calculate the outstanding balance at the end of 10 years.

Problem 51.10
The interest rate on a 30 year mortgage is 12% compounded monthly. Lauren is repaying the mortgage by paying monthly payments of 700. Additionally, to pay off the loan early, Lauren has made additional payments of 1000 at the end of each year.
Calculate the outstanding balance at the end of 10 years.

Problem 51.11
A mortgage loan is being repaid with level annual payments of 5000 at the end of the year for 20 years. The interest rate on the mortgage is 10% per year. The borrower pays 10 payments and then is unable to make payments for two years.
Calculate the outstanding balance at the end of the 12th year.

Problem 51.12
A 20,000 mortgage is being repaid with 20 annual installments at the end of each year. The borrower makes 5 payments, and then is temporarily unable to make payments for the next 2 years. Find an expression for the revised payment to start at the end of the 8th year if the loan is still to be repaid at the end of the original 20 years.

Measures of Interest Rate Sensitivity

In this chapter we study the sensitivity of the interest rate. We will cover the following topics:
(1) The effect of inflation on interest rate. We will consider the effect of inflation in the calculations of present values and accumulated values.
(2) The effect of the investment period on interest rates. That is, interest rate is sensitive to time. This fact is known as the **term structure of interest rate.** A plot of the interest rate of an investment against time to maturity will be discussed. We call such a plot a **yield curve.**
(3) We want to measure the change in a bond's price as interest rate change. We call this measure the **duration;** the two types of duration we will look at are the **modified duration** and the **Macaulay duration.**
(4) **Immunization** is a technique that helps investors shielding their overall financial position from exposure to interest rate fluctuations.
(5) **Absolute matching:** The idea here is to structure assets such that the resulting asset inflows will exactly match the liability outflow.

52 The Effect of Inflation on Interest Rates

What is inflation? **Inflation** is defined as a sustained increase in the general level of prices for goods and services. It is measured as an annual percentage increase, known as **rate of inflation**. As inflation rises, every dollar you own buys a smaller percentage of a good or service. That is, inflation represents a loss of purchasing power. The opposite to inflation is deflation.
Experience has shown that inflation has a significant effect on the rate of interest. As a matter of fact they are positively correlated in the sense that over time both tend to move in the same direction.
The following example illustrates how inflation affects the purchasing power of money. Suppose that today a gallon of milk costs \$4. Then with \$100 one can buy 25 gallons of milk. Suppose the \$100 is invested for 2 years at an 8% annual effective rate. Then at the end of the two years, you will have $100(1.08)^2 = \$116.64$. Without the presence of inflation, that is assuming the cost of milk is still \$4, you can now buy 29 gallons of milk. Now suppose that there has been a constant rate of inflation during the two years of 5% a year so that milk will cost $4(1.05)^2 = \$4.41$ per gallon. In this case, one can only buy $\frac{116.64}{4.41} = 26.45$ gallons of milk, a reduction of 3 gallons.
The real rate of return, i', is measured by solving the equation

$$25(1+i')^2 = \frac{116.64}{4.41} = 25\frac{(1.08)^2}{(1.05)^2}$$

by taking the square root of both sides we obtain

$$(1+i')(1.05) = 1.08$$

or

$$1 + i_{\text{real rate}} = \frac{1 + i_{nominal\ rate}}{1 + i_{inflation\ rate}}.$$

To study the effect of inflation on rates we introduce some notations. The rate of interest after eliminating the effect of inflation is called the **real rate of interest** and will be denoted by i', while the actual rate of interest charged in the market (i.e. the rate not adjusted for inflation) is called the **nominal interest rate** (not to be confused with the different meaning of nominal used in Section 9) and will be denoted by i. Let r denote the rate of inflation.
Assuming constant rate of inflation, the relationship between real and nominal interest rates can be described in the equation:

$$1 + i = (1+i')(1+r), \quad i > i' > 0, r > 0.$$

From this we can write

$$1 + i' = \frac{1+i}{1+r} \tag{52.1}$$

or

$$i' = \frac{i-r}{1+r}.$$

A common approximation for the real interest rate is given by the Fisher equation:

$$i' = i - r$$

which states that the real interest rate is the nominal interest rate minus the expected rate of inflation. If inflation is positive, which it generally is, then the real interest rate is lower than the nominal interest rate. That's the reason we are imposing the condition $i > i'$ whenever $r > 0$.
Formula (52.1) is quite useful in performing calculations involving rates of inflation. For example, assume that we wish to find the present value of a series of payments at the end of each period for n periods in which the base payment amount at time 0 is R, but each payment is indexed to reflect inflation. If r is the periodic rate of inflation and i is the periodic nominal interest rate, then

$$PV = R\left[\frac{1+r}{1+i} + \left(\frac{1+r}{1+i}\right)^2 + \cdots + \left(\frac{1+r}{1+i}\right)^n\right] = R(1+r)\frac{1-\left(\frac{1+r}{1+i}\right)^n}{i-r}.$$

However, if we use Formula (52.1) the above result becomes

$$PV = R\left[\frac{1}{1+i'} + \frac{1}{(1+i')^2} + \cdots + \frac{1}{(1+i')^n}\right] = Ra_{\overline{n}|i'}.$$

The above two formulas provide a guidance in computing present values of future payments as follows:
1. If future payments are not affected by inflation, then discount at the nominal rate of interest.
2. If future payments are adjusted to reflect the rate of inflation and the adjustment is reflected in the payment amount, then also discount at the nominal rate of interest.
3. If future payments are adjusted to reflect the rate of inflation but the adjustment is not reflected in the payment amount, the correct procedure is to discount at the real rate of interest.

We next consider inflation in connection with accumulated values. Consider the common situation in which an investor invests A dollars for n periods at interest rate i. The value of this investment in "nominal dollars" at the end of n periods is

$$AV = P(1+i)^n.$$

However, if the rate of inflation is accounted for, then the purchasing power of this investment is

$$AV = P\left(\frac{1+i}{1+r}\right)^n = P(1+i')^n.$$

Thus, the value of this investment in "real dollars" is lower since $i > i'$.

Example 52.1
Money is invested for 3 years at an interest rate of 4% effective. If inflation is 5% per year over this period, what percentage of purchasing power is lost?

Solution.
One dollar will accumulate to $(1.04)^3 = 1.1248$ over the period, but the purchasing power will only be $\frac{(1.04)^3}{(1.05)^3} = 0.9716$ due to inflation. The loss of purchasing power per dollar of original investment is $1 - 0.9716 = 2.84\%$ ■

Example 52.2
The real interest rate is 6% and the inflation rate is 4%. Allan receives a payment of 1,000 at time 1, and subsequent payments increase by 50 for 5 more years. Determine the accumulated value of these payments in nominal dollars at time 6 years.

Solution.
The nominal interest rate is:

$$i = (1.06)(1.04) - 1 = 10.24\%$$

The accumulated value of the cash flows is:

$$\begin{aligned} AV =& 1,000(1.1024)^5 + 1,050(1.1024)^4 + 1,100(1.1024)^3 \\ &+ 1,150(1.1024)^2 + 1,200(1.1024) + 1,250 = 8623.08 \end{aligned}$$ ■

Practice Problems

Problem 52.1
An insurance company is making annual payments under the settlement provisions of a personal injury lawsuit. A payment of $24,000 has just been made and ten more payments are due. Future payments are indexed to the Consumer Price Index which is assumed to increase at 5% per year. Find the present value of the remaining obligation if the assumed interest rate is 8%. Use the exact value of i'.

Problem 52.2
The nominal rate of interest is 10% and the rate of inflation is 5%. A single deposit is invested for 20 years. Let A denote the value of the investment at the end of 10 years, with the rate of inflation accounted for in the calculation. Let B denote the value of the investment at the end of 10 years computed at the real rate of interest. Find the ratio of $\frac{A}{B}$.

Problem 52.3
The nominal rate of interest is 8% and the rate of inflation is 5%. Calculate the real rate of interest.

Problem 52.4
Mindy invests 10,000 for 10 years at a nominal rate of interest of 8%. The rate of inflation is 5% over the 10 year period. Calculate the value at the end of 10 years of Mindy's investment in today's dollars.

Problem 52.5
As a result of sky-high gas prices, the cost of delivery pizza is expected to experience a constant 8% annual rate of inflation. In today's dollars, Obese Oliver spends $2,000 per year on delivery pizza. His delivery pizza-eating habits will not change over the next ten years, so he will pay whatever price increases are necessary to sustain this level of consumption. If the nominal interest rate (note that "nominal" refers to both interest convertibility and its relationship to inflation), convertible semiannually, is 10%, find the present value of Oliver's delivery pizza purchases for the next ten years.

Problem 52.6
Money is invested for 5 years in a savings account earnings 7% effective. If the rate of inflation is 10%, find the percentage of purchasing power lost during the period of investment.

Problem 52.7 ‡
An insurance company has an obligation to pay the medical costs for a claimant. Average annual claims costs today are $5,000, and medical inflation is expected to be 7% per year. The claimant

is expected to live an additional 20 years. Claim payments are made at yearly intervals, with the first claim payment to be made one year from today. Find the present value of the obligation if the annual interest rate is 5%.

Problem 52.8 ‡
Seth deposits X in an account today in order to fund his retirement. He would like to receive payments of 50 per year, in real terms, at the end of each year for a total of 12 years, with the first payment occurring seven years from now. The inflation rate will be 0.0% for the next six years and 1.2% per annum thereafter. The annual effective rate of return is 6.3%. Calculate X.

53 The Term Structure of Interest Rates and Yield Curves

In the previous section we discussed a factor affecting interest rates, i.e. inflation. In this section we discuss another factor which is the length of investment period.
The **term structure of interest rates** refers to the way interest rates depend on time. Usually long-term market interest rates are higher than short-term market interest rates. An example of term structure of interest rates is given in the table below.

Length of investment	Interest rate (or Spot rate)
1 year	3%
2 year	4%
3 year	6%
4 year	7%

Table 53.1

We can extend the values in this table to a continuous graph, and the resulting graphical illustration is called the **yield curve** corresponding to the table. The interest rates on the yield curve (or listed in the table) are often called **spot rates**.
Economists and investors believe that the shape of the yield curve reflects the market's future expectation for interest rates and the conditions for monetary policy. For example, an upward-sloping yield curve implies market's expectation of future increases in interest rates. In some cases, short-term rates exceed long-term rates, that is the yield curve has negative slope, in this case the curve is called **inverted.** When all maturities have the same yield the curve is called **flat**.
Now, recall from Section 30 the formula for the net present value of a series of future payments given by

$$NPV(i) = \sum_{t=0}^{n}(1+i)^{-t}c_t.$$

This formula is based on a single rate of interest i. With the presence of a term structure of interest rates, the net present value can be found by discounting each payment by its associated spot rate. That is, if we denote the spot rate for period t as i_t then the net present value is given by the equation

$$NPV(i_*) = \sum_{t=0}^{n}(1+i_t)^{-t}c_t.$$

This formula can be considered as an application of computing annuity values with varying rates of interest as discussed in Section 21.

Example 53.1
You are given the following term structure of interest rates:

Length of investment	Interest Rate
1	7.00%
2	8.00%
3	8.75%
4	9.25%
5	9.50%

Find the present value of payments of $1000 at the end of each year for five years using the spot rates given above. What level yield rate would produce an equivalent value?

Solution.
The present value of the payments is

$$1000[(1.07)^{-1} + (1.08)^{-2} + (1.0875)^{-3} + (1.0925)^{-4} + (1.095)^{-5}] = \$3906.63.$$

The equivalent level yield rate is found by solving the equation $a_{\overline{5}|i} = 3.90663$ which gives $i = 0.0883 = 8.83\%$ using a financial calculator ■

A special type of spot rates is the **forward interest rate**. In order to illustrate this concept, consider a firm which needs to borrow a sizeable amount of money for two years. The firm is presented with two options. The first option is to borrow money for 2 years at the 2-year spot rate of 10%. The second option is to borrow for one year at the one-year spot rate now of 8%, and to borrow for the second year at the one-year spot rate in effect a year later. This 1-year spot rate for the next year is an example of a **forward rate** or more specifically a **one year forward interest rate.** This is what the one year spot rate will be after one year is passed.
Now, if f is the forward rate then the firm will be indifferent between the two options if

$$(1.10)^2 = (1.08)(1 + f)$$

which gives $f = 0.12 = 12\%$. So, if the firm can predict that the forward rate will be greater than 12%, it should use the first option to borrow, but if they expect the forward rate to be less than 12%, it should use the second option to borrow.

Example 53.2
Consider the term structure of interest rates given in Table 53.1. Let f_n^{n+t} represents the $t-$year forward rate n years from now.
(a) Compute the 1-year, 2-year, and 3-year forward rate, one year from now.
(b) Compute the 1-year and 2-year forward rates two years from now.
(c) Compute the 1-year forward rate three years from now.

Solution.
(a) The 1-year forward rate one year from now satisfies the equation

$$(1.03)(1 + f_1^2) = (1.04)^2.$$

Solving this equation we find $f_1^2 = 5.01\%$.
The 2-year forward rate, one year from now satisfies the equation

$$(1.03)(1 + f_1^3)^2 = (1.06)^3.$$

Solving this equation we find $f_1^3 = 7.53\%$.
The 3-year forward rate, one year from now satisfies the equation

$$(1.03)(1 + f_1^4)^3 = (1.07)^4.$$

Solving this equation we find $f_1^4 = 8.37\%$.
(b) The 1-year forward rate, two years from now satisfies the equation

$$(1.04)^2(1 + f_2^3) = (1.06)^3.$$

Solving this equation we find $f_2^3 = 10.1\%$.
The 2-year forward rate, two years from now satisfies the equation

$$(1.04)^2(1 + f_2^4)^2 = (1.07)^4.$$

Solving this equation we find $f_2^4 = 10.00\%$.
(c) The 1-year forward rate three years from now satisfies the equation

$$(1.06)^3(1 + f_3^4) = (1.07)^4.$$

Solving this equation we find $f_3^4 = 10.06\%$ ■

From any yield curve, one can calculate a complete set of implied forward rates. In general, the $k-$year forward rate n years from now satisfies the equation

$$(1 + i_n)^n(1 + f_n^{n+k})^k = (1 + i_{n+k})^{n+k}$$

where i_t is the $t-$year spot rate. Note that f_n^{n+k} is the forward interest rate between the periods n and $n + k$.

Example 53.3
Find the present value of the remaining payments in the annuity immediate given in Example 53.1 immediately after two payments have been made assuming that $f_2^{2+t} = 1\% + i_t$ where $t = 1, 2, 3$.

Solution.
The comparison date for this calculation is at the end of two years. At that time there are three remaining annuity payments to be made. The expected forward rates are the current spot rates for 1,2, and 3 years each increased by 1%. Thus, the present value of the remaining three payments is

$$1000[(1.08)^{-1} + (1.09)^{-2} + (1.0975)^{-3}] = \$2524.07 \blacksquare$$

Example 53.4
You are given the following term of structure of spot interest rates:

Term (in Years)	Spot Interest Rate
1	7.00%
2	8.00%
3	8.75%
4	9.25%
5	9.50%

(a) Find the price of a \$1000 two-year bond with annual 5% coupon using the spot rates given above.
(b) Compute the yield to maturity as defined in Section 43.

Solution.
(a) The coupon is \$50 each. The price of the bond is

$$P = 50(1.07)^{-1} + 1050(1.08)^{-2} = 946.93$$

(b) The yield to maturity i satisfies the equation

$$50(1+i)^{-1} + 1050(1+i)^{-2} = 946.93.$$

Solving this equation we find $i = 7.975\%$ ■

Example 53.5
Yield rates for 6% annual coupon bonds are given to be 3.5% for 1-year bonds and 4% for 2-year bonds. Find the implied 2-year spot rate from these yield rates.

Solution.
Per 100 of par value we have

$$6a_{\overline{2}|0.04} + 100v_{0.04}^2 = \frac{6}{1.035} + \frac{106}{(1+i)^2}.$$

Solving this equation for i we find $i = 4.01\%$ ■

Practice Problems

Problem 53.1
You are given the following term structure of interest rates:

Length of investment	Interest Rate
1	5.00
2	6.00
3	6.75
4	7.25
5	7.50

Table 53.2

Which of the following are true:

(I) The yield curve has a positive slope.
(II) The yield curve is inverted.
(III) The interest rates in the table are called forward rates.

Problem 53.2
Using Table 53.2, calculate the present value of a five year annuity due of 100 per year.

Problem 53.3
Using Table 53.2, determine the three year deferred two year forward rate.

Problem 53.4
Using Table 53.2, determine the accumulated value of a three year annuity immediate of 100 at the end of each year.

Problem 53.5
You are given the following yield curve

Length of investment	Interest Rate
1	0.040
2	0.045
3	0.048
4	0.050
5	0.051

(a) Calculate the present value of an annuity immediate of 10 at the end of each year for 5 years using the spot rates.
(b) Calculate the equivalent level rate.

Problem 53.6
You are given the following yield curve

Length of investment	Interest Rate
1	0.040
2	0.045
3	0.048
4	0.050
5	0.051

(a) Calculate the present value of an annuity due of 10 at the beginning of each year for 5 years using the spot rates.
(b) Calculate the equivalent level rate.

Problem 53.7
You are given the following yield curve

Length of investment	Interest Rate
1	0.040
2	0.045
3	0.048
4	0.050
5	0.051

(a) Calculate the 2 year deferred 3 year forward rate less the three year spot rate.
(b) Calculate what the 3 year spot rate is expected to be in 2 years.

Problem 53.8
You are given the following term structure of spot interest rates:

Term (in Years)	Spot Interest Rate
1	7%
2	8%
3	9%

What is the one-year forward rate beginning two years from now?

Problem 53.9 ‡
Using the table of the previous problem, find
(a) the current price of a \$1000 3-year bond with coupon rate 6% payable annually.
(b) the annual effective yield rate for the bond if the bond is sold at a price equal to its value.

Problem 53.10 ‡
You are given the following term structure of spot interest rates:

Term (in Years)	Spot Interest Rate
1	5.00%
2	5.75%
3	6.25%
4	6.50%

A three-year annuity-immediate will be issued a year from now with annual payments of 5000. Using the forward rates, calculate the present value of this annuity a year from now.

Problem 53.11 ‡
Consider a yield curve defined by the following equation:

$$i_k = 0.09 + 0.002k - 0.001k^2$$

where i_k is the annual effective rate of return for zero coupon bonds with maturity of k years. Let j be the one-year effective rate during year 5 that is implied by this yield curve. Calculate j.

Problem 53.12 ‡
Yield rates to maturity for zero coupon bonds are currently quoted at 8.5% for one-year maturity, 9.5% for two-year maturity, and 10.5% for three-year maturity. Let i be the one-year forward rate for year two implied by current yields of these bonds. Calculate i.

Problem 53.13
You are given the following term of structure of spot interest rates:

Term (in Years)	Spot Interest Rate
1	7.00%
2	8.00%
3	8.75%
4	9.25%
5	9.50%

Find the following expected forward rates:
(a) 1-year deferred 2-year forward rate.
(b) 2-year deferred 3-year forward rate.

Problem 53.14
You are given the following term of structure of spot interest rates:

Term (in Years)	Spot Interest Rate
1	7.00%
2	8.00%
3	8.75%
4	9.25%
5	9.50%

(a) Find the price of a \$1000 two-year bond with annual 10% coupon using the spot rates given above.
(b) Compute the yield to maturity as defined in Section 41.

Problem 53.15
You are given $1-$year, $2-$year, and $3-$year spot rates of 4%, 5%, and 6%, respectively. Calculate the annual yield rate for $3-$year 5% annual coupon bonds implied by these spot rates.

Problem 53.16
Consider a yield curve defined by the following equation:

$$i_k = 0.09 + 0.002k - 0.001k^2$$

where i_k is the annual effective rate of return for zero coupon bonds with maturity of k years.
(a) Calculate the 2-year spot rate implied by this yield curve.
(b) Calculate the 2-year deferred 3-year forward rate implied by this yield curve.

Problem 53.17
A two-year bond with annual coupons of 100 and redemption value of 1000 is priced at 1,037.41. The current two-year spot rate is 8%. Determine the current one-year spot rate that is consistent with the pricing of the bond.

Problem 53.18
A 5 year bond with 6% annual coupons has a yield rate of 10% effective and a 5 year bond with 8% annual coupons has a yield rate of 9% effective. What is the 5 year spot rate?

Problem 53.19 ‡
Which of the following statements about zero-coupon bonds are true?
(I) Zero-coupon bonds may be created by separating the coupon payments and redemption values from bonds and selling each of them separately.
(II) The yield rates on stripped Treasuries at any point in time provide an immediate reading of the risk-free yield curve.
(III) The interest rates on the risk-free yield curve are called forward rates.

54 Macaulay and Modified Durations

By now, it should be clear to you that the timing of cash flows is a significant factor in the analysis of financial instruments. In this section we develop some "indices" to measure the timing of future payments. One index known as the **duration** is useful as a measure of the sensitivity of a bond's price to interest rate movements. The two types of duration we will look at are the **modified duration** and the **Macaulay duration.**
The most basic index of time measuring is the **term-to-maturity.** For a given price and par value, among zero coupon bonds, one wishes to purchase the bond with the earliest redemption time. In the case of bonds carrying coupons, the redemption time is not enough for helping choose the bond to purchase.
A better index than the term-to-maturity is the **method of equated time**, which was defined in Section 13:
Let $R_1, R_2, \cdots, R_n$ be a series of payments made at times $1, 2, \cdots, n$. Then the weighted average of the various times of payments, where the weights are the various amounts paid is

$$\bar{t} = \frac{\sum_{t=1}^{n} tR_t}{\sum_{t=1}^{n} R_t}.$$

This is also known as the **average term-to-maturity.**

Example 54.1
Consider two 10-year par value 100 bonds one with 5% coupons and the other with 10% coupons. Find the equated time for each coupon.

Solution.
The average term to maturity of the 5% bond is

$$\bar{t} = \frac{1 \cdot 5 + 2 \cdot 5 + \cdots + 10 \cdot 5 + 10 \cdot 100}{5 + 5 + \cdots + 5 + 100} = 8.50.$$

The average term to maturity of the 10% bond is

$$\bar{t} = \frac{1 \cdot 10 + 2 \cdot 10 + \cdots + 10 \cdot 10 + 10 \cdot 100}{10 + 10 + \cdots + 10 + 100} = 7.75.$$

This shows that the 5% bond is a longer term bond than the 10% bond ■

An even better index that is similar to the method of equated time is obtained by replacing each payment in the above formula by its present value thus obtaining

$$\bar{d} = \frac{\sum_{t=1}^{n} t\nu^t R_t}{\sum_{t=1}^{n} \nu^t R_t}.$$

We call $\overline{d}$ the **Macaulay duration** or simply **duration**.

Example 54.2
Suppose payments of 2000, 4000, and 10000 are to be made at times 1, 2, and 4, respectively. Assume, an annual yield of 25%.
(a) Find the average term to maturity using the method of equated time.
(b) Find the duration of the investment.

Solution.
(a) The method of equated time produces the value

$$\bar{t} = \frac{1\cdot 2000 + 2\cdot 4000 + 4\cdot 10000}{2000+4000+10000} = 3.125.$$

(b) The duration is given by

$$\overline{d} = \frac{1\cdot(1.25)^{-1}\cdot 2000 + 2\cdot(1.25)^{-2}\cdot 4000 + 4\cdot(1.25)^{-4}\cdot 10000}{(1.25)^{-1}(2000)+(1.25)^{-2}(4000)+(1.25)^{-4}(10000)} = 2.798 \blacksquare$$

Example 54.3
Consider two bonds purchased at the redemption value of 1000, and due in 5 years. One bond has 5% annual coupon rate payable semi-annually and the other has 10% annual coupon rate payable semi-annually. Find the duration of each bond if both bonds were purchased to yield 7% compounded semi-annually.

Solution.
The duration of the 5% bond is

$$\begin{aligned}
\overline{d} =& \frac{0.5[1\cdot(1.035)^{-1}(25)+2\cdot(1.035)^{-2}(25)+\cdots+10\cdot(1.035)^{-10}(25)]+5000(1.035)^{-10}}{(1.035)^{-1}(25)+(1.035)^{-2}(25)+\cdots+(1.035)^{-10}(25)+1000(1.035)^{-10}}\\
=& \frac{12.5(Ia)_{\overline{10}|0.035}+5000(1.035)^{-10}}{25a_{\overline{10}|0.035}+1000(1.035)^{-10}}\\
=& 4.4576
\end{aligned}$$

The duration of the 10% bond is

$$\begin{aligned}
\overline{d} =& \frac{0.5[1\cdot(1.035)^{-1}(50)+2\cdot(1.035)^{-2}(50)+\cdots+10\cdot(1.035)^{-10}(50)]+5000(1.035)^{-10}}{(1.035)^{-1}(50)+(1.035)^{-2}(50)+\cdots+(1.035)^{-10}(50)+1000(1.035)^{-10}}\\
=& 4.1158
\end{aligned}$$

Note that $\overline{d}$ is a function of i. If $i = 0$ we have $\overline{d} = \overline{t}$ so that the method of equated time is a special case of duration which ignores interest.
Also, note that in the case of only one future payment, duration is the point in time at which that payment is made. This is clear since the summations in the numerator and the denominator have only one term each and everything cancels except the time of payment.

Example 54.4
Find the duration of a 10-year zero coupon bond assuming a yield 8% effective.

Solution.
Only one payment is involved so that $\overline{d} = 10$. Notice that this answer is independent of the yield rate ■

Another important fact is that $\overline{d}$ is a decreasing function of i. To see this, we have

$$\begin{aligned}\frac{d}{di}(\overline{d}) =& \frac{d}{di}\left(\frac{\sum_{t=1}^n t\nu^t R_t}{\sum_{t=1}^n \nu^t R_t}\right)\\ =& -\nu\frac{\left(\sum_{t=1}^n \nu^t R_t\right)\left(\sum_{t=1}^n t^2\nu^t R_t\right) - \left(\sum_{t=1}^n t\nu^t R_t\right)^2}{\left(\sum_{t=1}^n \nu^t R_t\right)^2}\\ =& -\nu\left[\frac{\sum_{t=1}^n t^2\nu^t R_t}{\sum_{t=1}^n \nu^t R_t} - \left(\frac{\sum_{t=1}^n t\nu^t R_t}{\sum_{t=1}^n \nu^t R_t}\right)^2\right]\end{aligned}$$

Now using Cauchy-Schwartz's inequality

$$\left(\sum_{t=1}^n a_t b_t\right)^2 \le \left(\sum_{t=1}^n a_t^2\right)\left(\sum_{t=1}^n b_t^2\right)$$

with

$$a_t = \nu^{\frac{t}{2}} R_t^{\frac{1}{2}} \text{ and } b_t = t\nu^{\frac{t}{2}} R_t^{\frac{1}{2}}$$

we find that

$$\frac{\sum_{t=1}^n t^2\nu^t R_t}{\sum_{t=1}^n \nu^t R_t} - \left(\frac{\sum_{t=1}^n t\nu^t R_t}{\sum_{t=1}^n \nu^t R_t}\right)^2 \ge 0$$

and therefore the derivative of $\overline{d}$ with respect to i is negative.

We next describe a measure of how rapidly the present value of a series of future payments changes as the rate of interest changes. Let $P(i)$ denote the present value of all future payments. Then by Taylor series we can write

$$P(i+\epsilon) = P(i) + \frac{\epsilon}{1!}P'(i) + \frac{\epsilon^2}{2!}P''(i) + \cdots.$$

Using the first two terms we can write

$$P(i+\epsilon) \approx P(i) + \epsilon P'(i).$$

Hence, the percentage change in P is given by

$$\frac{P(i+\epsilon) - P(i)}{P(i)} \approx \epsilon \frac{P'(i)}{P(i)}.$$

Thus, the percentage change of $P(i)$ is a function of $\frac{P'(i)}{P(i)}$. We define the **volatility** of the present value by

$$\overline{\nu} = -\frac{P'(i)}{P(i)} = -\frac{d}{di}[\ln P(i)].$$

The minus sign is necessary to make $\overline{\nu}$ positive since $P'(i)$ is negative (increasing the yield rate results in a decreasing present value).
Now, since $P(i) = \sum_{t=1}^{n}(1+i)^{-t}R_t$, by taking the derivative we find $P'(i) = -\sum_{t=1}^{n} t(1+i)^{-t-1}R_t$. Hence,

$$\overline{\nu} = \frac{\sum_{t=1}^{n} t(1+i)^{-t-1}R_t}{\sum_{t=1}^{n}(1+i)^{-t}R_t} = \frac{1}{1+i}\cdot\frac{\sum_{t=1}^{n} t(1+i)^{-t}R_t}{\sum_{t=1}^{n}(1+i)^{-t}R_t} = \frac{\overline{d}}{1+i} = \nu\overline{d}.$$

Because of this close relationship, volatility is often called **modified duration.**

Remark 54.1
None of the results in this section are valid if the payment amounts vary depending on the interest rate.

Example 54.5
(a) Find the Macaulay duration of a ten-year, \$1,000 face value, 8% annual coupon bond. Assume an effective annual interest rate of 7%.
(b) Find the modified duration for the bond

Solution.
(a) We have

$$\begin{aligned}\overline{d} =& \frac{80\nu + 2(80\nu^2) + \cdots + 10(80\nu^{10}) + 10(1000\nu^{10})}{80\nu + (80\nu^2) + \cdots + (80\nu^{10}) + (1000\nu^{10})}\\ =& \frac{80(Ia)_{\overline{10}|0.07} + 10000\nu_{0.07}^{10}}{80a_{\overline{10}|0.07} + 1000\nu_{0.07}^{10}}\end{aligned}$$

But

$$\begin{aligned}
\nu_{0.07}^{10} =&(1.07)^{-10} = 0.508349\\
a_{\overline{10}|0.07} =&\frac{1-(1.07)^{-10}}{0.07} = 7.02358\\
(Ia)_{\overline{10}|0.07} =&\frac{1.07a_{\overline{10}|0.07} - 10(1.07)^{-10}}{0.07} = 34.7391
\end{aligned}$$

Hence,

$$\overline{d} = \frac{80(34.7392)+10000(0.508349)}{80(7.02358)+1000(0.508349)} = 7.3466 \text{ years}$$

(b) The modified duration is

$$\overline{\nu} = \nu\overline{d} = (1.07)^{-1}(7.3466) = 6.8660 \blacksquare$$

Example 54.6
A 10-year \$100,000 mortgage will be repaid with level semi-annual payments of interest and principal. Assume that the interest rate on the mortgage is 8%, convertible semiannually, and that payments are made at the end of each half-year. Find the Macaulay duration of this mortgage.

Solution.
Per dollar of mortgage payment, the Macaulay duration is

$$\begin{aligned}
\overline{d} =&0.5\frac{\nu_{0.04} + 2\nu_{0.04}^2 + 3\nu_{0.04}^3 + \cdots + 20\nu_{0.04}^{20}}{\nu_{0.04} + \nu_{0.04}^2 + \nu_{0.04}^3 + \cdots + \nu_{0.04}^{20}}\\
=&\frac{0.5(Ia)_{\overline{20}|0.04}}{a_{\overline{20}|0.04}} = 4.6046 \text{ years}
\end{aligned}$$

Note that the answer is independent of the rate of interest being paid on the mortgage. Duration depends on the pattern of the level payments and not their amount $\blacksquare$

Example 54.7
Find the duration of a preferred stock paying level dividends into perpetuity assuming effective 8%.

Solution.
Per dollar dividend we have

$$\overline{d} = \frac{(Ia)_{\overline{\infty}|}}{a_{\overline{\infty}|}} = \frac{1.08/.08^2}{1/.08} = 13.5 \blacksquare$$

Example 54.8
Find the modified duration for a share of common stock. Assume that the stock pays annual dividends, with the first dividend of \$2 payable 12 months from now, and that subsequent dividends will grow at an annual rate of 4%. Assume that the effective annual interest rate is 9%.

Solution.
We have

$$\begin{aligned}P(i) =&\frac{D_1}{i-g}\\ P'(i) =&-D_1(i-g)^{-2}\\ \overline{v} =&-\frac{P'(i)}{P(i)}=\frac{1}{i-g}=\frac{1}{0.09-0.04}=20 \blacksquare\end{aligned}$$

Example 54.9
You are the actuary for an insurance company. Your company's liabilities include loss reserves, which are liability reserves set aside to make future claim payments on policies which the company has already sold. You believe that these liabilities, totaling \$100 million on December 31, 2006, will be paid out according to the following schedule:

Calendar year	Proportionl of reserves paid out
2007	40%
2008	30%
2009	20%
2010	10%

Find the modified duration of your company's loss reserves. Assume that the annual interest rate is 10%, and that all losses paid during a given calendar year are paid at the mid-point of that calendar year.

Solution.
We have

$$\overline{\nu}=\nu\left(\frac{0.5(4\nu^{0.5})+1.5(3\nu^{1.5})+2.5(2\nu^{2.5})+3.5(\nu^{3.5})}{4\nu^{0.5}+3\nu^{1.5}+2\nu^{2.5}+\nu^{3.5}}\right)=1.2796 \blacksquare$$

Remark 54.2
The equation

$$\frac{P(i+\epsilon)-P(i)}{P(i)}=-\epsilon\overline{\nu}$$

is valid if $P(i)$ stands for the current price of a bond.

Example 54.10
The current price of a bond is \$110 and its yield is 7%. The modified duration is 5. Estimate the price of the bond if its yield falls to 6%.

Solution.
The approximate percentage change in the price is $-(-0.01)(5) = 5\%$. That is, the price of the bond increased by approximately 5%. Thus, the new price of the bond is $110 \times 1.05 = \$115.50$ ■

Portfolio Modified Duration
Consider a portfolio consisting of n bonds. Let bond k have current price $P_k(i)$ and modified duration $\overline{\nu}_k(i)$. Then the current value of the portfolio is

$$P(i) = P_1(i) + P_2(i) + \cdots + P_n(i).$$

Let $\overline{\nu}$ denote the modified duration of the portfolio. Then, we have

$$\begin{aligned}\overline{\nu} = & -\frac{P'(i)}{P(i)} = -\frac{P_1'(i)}{P(i)} + \left(-\frac{P_2'(i)}{P(i)}\right) + \cdots + \left(-\frac{P_n'(i)}{P(i)}\right)\\ =&\frac{P_1(i)}{P(i)}\left(-\frac{P_1'(i)}{P_1(i)}\right) + \frac{P_2(i)}{P(i)}\left(-\frac{P_2'(i)}{P_2(i)}\right) + \cdots + \frac{P_n(i)}{P(i)}\left(-\frac{P_n'(i)}{P_n(i)}\right)\\ =&\frac{P_1(i)}{P(i)}\overline{\nu}_1 + \frac{P_2(i)}{P(i)}\overline{\nu}_2 + \cdots + \frac{P_n(i)}{P(i)}\overline{\nu}_n\end{aligned}$$

Thus, the modified duration of a portfolio can be calculated as the weighted average of the bonds' modified durations, using the market values of the bonds as the weights.

Example 54.11
Megan buys the following bonds:
(I) Bond A with a modified duration of 7 for 1000;
(II) Bond B with a modified duration of 5 for 2000; and
(III) Bond C with a modified duration of 10 for 500.
Calculate the modified duration of the portfolio.

Solution.
The modified duration of the portfolio is weighted average of the modified durations weighted by the purchase price

$$\overline{\nu} = \frac{7 \cdot 1000 + 5 \cdot 2000 + 10 \cdot 500}{1000 + 2000 + 500} = 6.286 \blacksquare$$

Example 54.12
An investment portfolio consists of two bonds: a two-year zero-coupon bond, and a four-year zero-coupon bond. Both bonds have redemption values of \$1,000. Assume an effective annual interest rate of 8%. Find the Macaulay duration of the investment portfolio.

Solution.
The Macaulay duration of the two-year zero-coupon is 2 with current value $1000v^2$. The Macaulay duration of the four-year zero-coupon is 4 with current value $1000v^4$. The Macaulay duration of the portfolio is

$$\overline{d} = \frac{2(1000\nu^2) + 4(1000\nu^4)}{1000\nu^2 + 1000\nu^4} = 2.9232 \text{ years} \blacksquare$$

Practice Problems

Problem 54.1
Find the modified duration of a 30-year, $1,000 face value, 6% annual coupon bond. Assume an effective annual interest rate of 9%.

Problem 54.2
Calculate the duration of a payment to be made in 5 years.

Problem 54.3
A 10 year bond has annual coupons of 10 and matures for 100. Which of the following are true:
(I) The term to maturity is 10 years.
(II) The average term to maturity using the method of equated time is 7.5 years.
(III) The Macaulay duration at 8% interest is 6.97.
(IV) The modified duration at 8% interest is 6.45.

Problem 54.4
Which of the following are true:
(I) The average term to maturity under the method of equated time is always greater than the volatility.
(II) The duration is a decreasing function of i.
(III) A zero coupon bond will have a Macaulay duration equal to the term to maturity.

Problem 54.5
(a) Calculate the duration of a 12-year annuity immediate payable using an interest rate of 5%.
(b) Calculate the modified duration of the annuity.

Problem 54.6
A perpetuity pays 100 immediately. Each subsequent payment is increased by inflation. Calculate the duration of the perpetuity using 10.25% assuming that inflation will be 5% annually.

Problem 54.7
Calculate the duration of a perpetuity immediate of 1 less the duration of a perpetuity due of 1 at an interest rate of 10%.

Problem 54.8
Calculate the modified duration of an annuity due with payments of 100 for 10 years using an interest rate of 8%.

Problem 54.9 ‡
A bond will pay a coupon of 100 at the end of each of the next three years and will pay the face value of 1000 at the end of the three-year period. The bond's duration (Macaulay duration) when valued using an annual effective interest rate of 20% is X. Calculate X.

Problem 54.10
Suppose that a 3-year financial instrument is expected to make increasing payments to you at the end of each of the next three years. Specifically, the payments will be $CF(t) = 1,000t$, for $t = 1, 2,$ and 3. Assume that you purchase this financial instrument, at time 0, at a price which provides an annual effective yield of 8%. Calculate the Macaulay duration, and the modified duration of this financial instrument.

Problem 54.11
Calculate the Macaulay duration and the modified duration of a 30-year zero-coupon bond with a face value of \$ 1,000. Assume that the annual yield-to-maturity is 8%.

Problem 54.12
A 10-year \$200,000 mortgage will be paid off with level quarterly amortization payments. Assume that the interest rate on the mortgage is 6%, convertible quarterly, and that payments are made at the end of each quarter. Find the modified duration of this mortgage.

Problem 54.13 ‡
Calculate the duration of a common stock that pays dividends at the end of each year into perpetuity. Assume that the dividend increases by 2% each year and that the effective rate of interest is 5%.

Problem 54.14 ‡
Calculate the duration of a common stock that pays dividends at the end of each year into perpetuity. Assume that the dividend is constant, and that the effective rate of interest is 10%.

Problem 54.15 ‡
The current price of an annual coupon bond is 100. The derivative of the price of the bond with respect to the yield to maturity is -700. The yield to maturity is an annual effective rate of 8%. Calculate the duration of the bond.

Problem 54.16 ‡
Calculate the Macaulay duration of an eight-year 100 par value bond with 10% annual coupons and an effective rate of interest equal to 8%.

Problem 54.17 ‡
John purchased three bonds to form a portfolio as follows:
Bond A has semiannual coupons at 4%, a duration of 21.46 years, and was purchased for 980.
Bond B is a 15-year bond with a duration of 12.35 years and was purchased for 1015.
Bond C has a duration of 16.67 years and was purchased for 1000.
Calculate the duration of the portfolio at the time of purchase.

Problem 54.18
A 10-year 1000 face value 6% annual coupon bond with redemption value C has duration equal to 6.06 using an annual effective interest rate of 6%. Calculate C.

Problem 54.19
You are given:
(i) the annual yield rate on a zero-coupon bond with duration of 6 months is 3%
(ii) the annual yield rate on a zero-coupon bond with duration of 12 months is 3.25%
(iii) the annual yield rate on a zero-coupon bond with duration of 18 months is 3.5%
(iv) the annual yield rate on a zero-coupon bond with duration of 24 months is 3.75%
Determine the semiannual effective yield rate on a 2-year 100 par value bond with 4% semiannual coupons.

Problem 54.20
John purchased three bonds to form a portfolio as follows:
Bond A is purchased for X and has a duration of 20 years.
Bond B is purchased for Y and has a duration of 20 years.
Bond C is purchased for $X + Y$.
The duration (in years) of the portfolio at the time of purchase is 18. Determine the duration (in years) of Bond C at the time of purchase.

55 Redington Immunization and Convexity

Most financial institutions try to insulate their portfolios from interest rate risk movements. Some institutions (such as banks) are concerned with protecting their current net worth against interest rate movements. Other institutions (such as pension funds) may face an obligation to make payments after a given number of years. These institutions are more concerned with protecting the future values of their portfolios.
The common factor amongst different investors and financial institutions is the presence of interest rate risk because net worth fluctuates with interest rate. Financial institutions try to structure their assets and liabilities in such a way that they are protected against small changes in interest rates.
There are three approaches that all work toward the same goal of minimizing (or even eliminating) the risk caused by fluctuating interest rates: Immunizatiom, full immunization, and dedication. In this section, we consider the first strategy.
The basic idea of **immunization** is that financial institutions (i.e., banks) are vulnerable to incurring losses if interest rates change. As a result, they "immunize" themselves by structuring their assets and liabilities such that the modified duration of the assets equals the modified duration of the liabilities. When this occurs, a rise in the interest rate would cause the present value of assets to decline, but there would be a matching decrease in the present value of liabilities. Thus, the company is protected against small changes in interest rates.
We next discuss how immunization achieves the protection required. Consider cash inflows $A_1, A_2, \cdots, A_n$ generated by the assets at times $1, 2, \cdots, n$. Similarly, let $L_1, L_2, \cdots, L_n$ be the cash outflows generated by the liabilities at times $1, 2, \cdots, n$. Let the net cash flows at time t be

$$R_t = A_t - L_t, \quad t = 1, 2, \cdots, n.$$

In general, banks assets and liabiltiy are roughly equal in size so that we can assume that the present value of the cash inflows from the assets is equal to the present value of the cash outflows from the liabilities. This leads to

$$P(i) = \sum_{t=1}^{n} \nu^t R_t = 0.$$

Next, we would like $P(i)$ to have a local minimum at i, so that small changes in the interest rate in either direction will increase the present value of the cash flows. But for the function $P(i)$ to have a minimum requires two conditions to hold. The first is that $P'(i) = 0$ (i.e. the modified duration of the assets is equal to the modified duration of the liabilities). The second condition is $P''(i) > 0$.
The second derivative of $P(i)$ is used to define the **convexity** of a series of cash flows given by

$$\overline{c} = \frac{P''(i)}{P(i)}.$$

To summarize, immunization is achieved when the following three conditions are met:
(1) $P(i) = 0$: The present value of cash inflows (assets) should be equal to the present value of cash outflows (liabilities).
(2) $P'(i) = 0$: The modified duration of the assets is equal to the modified duration of the liabilities.
(3) $P''(i) > 0$: The convexity of the present value of cash inflows (assets) should be greater than the convexity of the present value of cash outflows(liabilities). In other words, asset growth (decline) should be greater (less) than liability growth(decline).
In practice there are some difficulties and limitations in implementing the immunization strategy:
(a) choice of i is not always clear.
(b) doesn't work well for large changes in i. For large changes in i a technique, known as **full immunization**, was developed for that purpose. This concept will be covered in Section 56.
(c) yield curve is assumed to change with the change in i; actually, short-term rates are more volatile than long-term rates.
(d) frequent rebalancing is required in order to keep the modified duration of the assets and liabilities equal.
(e) exact cash flows may not be known and may have to be estimated.
(f) assets may not have long enough maturities of duration to match liabilities.

We next express the percentage change of $P(i)$ in terms of both the modified duration and convexity. Expanding $P(i+\epsilon)$ as a Taylor series as far as second derivatives we find

$$P(i+\epsilon) \approx P(i) + \epsilon P'(i) + \frac{\epsilon^2}{2} P''(i).$$

Thus,

$$\frac{P(i+\epsilon) - P(i)}{P(i)} \approx \epsilon \frac{P'(i)}{P(i)} + \frac{\epsilon^2}{2} \frac{P''(i)}{P(i)} = -\epsilon \overline{\nu} + \frac{\epsilon^2}{2} \overline{c}.$$

Example 55.1
A client deposits 100,000 in a bank, with the bank agreeing to pay 8% effective for two years. The client indicates that half of the account balance will be withdrawn at the end of the first year. The bank can invest in either one year or two year zero coupon bonds. The one year bonds yield 9% and the two year bonds yield 10%. Develop an investment program based on immunization.

Solution.
The cash outflows are $50000(1.08)$ at the end of the first year and $50000(1.08)^2$ at the end of the second year.

Suppose the bank invests x in the one year bonds, and y in two year bonds. Then

$$\begin{aligned}P(i) =&x(1.09)(1+i)^{-1} + y(1.10)^2(1+i)^{-2} - 50000(1.08)(1+i)^{-1}\\&-50000(1.08)^2(1+i)^{-2}\\P'(i) =&-x(1.09)(1+i)^{-2} - 2(1.10)^2y(1+i)^{-3} + 50000(1.08)(1+i)^{-2}\\&+2(50000)(1.08)^2(1+i)^{-3}\\P''(i) =&2(1.09)x(1+i)^{-3} + 6(1.10)^2y(1+i)^{-4} - 100000(1.08)(1+i)^{-3}\\&-300000(1.08)^2(1+i)^{-4}\end{aligned}$$

To immunize against interest rate risk at $i = 0.08$, the bank should have $P(0.08) = 0, P'(0.08) = 0$ and $P''(0.08) > 0$. The first two conditions lead to a system in the unknowns x and y. Solving the system gives $x = 49541.28$ and $y = 48198.35$. Moreover, $P''(0.08) = 128,773.89 > 0$. Thus, the bank is well protected against interest rate fluctuations ■

Example 55.2
For the assets in Example 55.1, find the modified duration and convexity.

Solution.
We have $P_A(i) = x(1.09)(1+i)^{-1} + (1.10)^2y(1+i)^{-2} \to P_A(0.08) = 50,000.00 + 50,000.00 = 100,000.00$. Also, $P_A'(i) = -x(1.09)(1+i)^{-2} - 2(1.10)^2y(1+i)^{-3} \to P_A'(0.08) = -138,888.89$. The modified duration is

$$\overline{v}(0.08) = -\frac{P_A'(0.08)}{P_A(0.08)} = \frac{138,888.89}{100,000.00} = 1.39.$$

The convexity is

$$\overline{c}(0.08) = \frac{P_A''(0.08)}{P_A(0.08)} = \frac{342,935.54}{100,000.00} = 3.43$$

since $P_A''(i) = 2(1.09)x(1+i)^{-3} + 6(1.10)^2y(1+i)^{-4} \to P_A''(0.08) = 342,935.54$ ■

Example 55.3
For a 30-year home mortgage with level payments and an interest rate of 10.2% convertible monthly, find the modified duration and the convexity of the payments.

Solution.
The monthly rate of interest is $0.102/12 = 0.0085$. Then for a dollar of monthly payment, we have

$$P(0.0085) = \sum_{t=1}^{360}(1.0085)^{-t} = (1.0085)^{-1}\left[\frac{1-(1.0085)^{-360}}{1-(1.0085)^{-1}}\right] = 112.0591$$

and

$$P'(0.0085) = -\sum_{t=1}^{360} t(1.0085)^{-t-1} = -(1.0085)^{-1}(Ia)_{\overline{360}|0.0085} = -11,188.69608$$

Thus, the modified duration is

$$\overline{\nu}(0.0085) = \frac{11,188.69608}{112.0591} = 99.85.$$

Thus, the modified duration of the payments is just under 100 months of the 360-month term of the mortgage.
Now,

$$\begin{aligned}P''(0.0085) =& \sum_{t=1}^{360} t(t+1)(1.0085)^{-t-2} = \nu^2 \sum_{t=1}^{360}(t^2+t)\nu^t \\ =& \nu^2[\sum_{t=1}^{360} t^2\nu^t + \sum_{t=1}^{360} t\nu^t] \\ =& \nu^2[\sum_{t=1}^{360} t^2\nu^t + (Ia)_{\overline{360}|0.0085}]\end{aligned}$$

Hence, using a calculator we find

$$\overline{c}(0.0085) = \frac{1,940,079 + 11,283.80}{(1.0085)^2(112.0591)} = 17,121 \blacksquare$$

Example 55.4
A bank agrees to pay 5% compounded annually on a deposit of 100,000 made with the bank. The depositor agrees to leave the funds on deposit on these terms for 8 years. The bank can either buy 4 year zero coupon bonds or preferred stock, both yielding 5% effective. How should the bank apportion its investment in order to immunize itself against interest rate risk?

Solution.
The present value of the cash flows in terms of the amount B invested in the bonds is

$$P(i) = B(1.05)^4(1+i)^{-4} + \frac{(100000 - B)(0.05)}{i} - 100000(1.05)^8(1+i)^{-8}$$

since the preferred stock is assumed to pay 5% forever and the bank must repay the deposit with interest at the end of 8 years.
Substitution verifies that $P(.05) = 0$. On the other hand,

$$P'(0.05) = -4B(1.05)^4(1.05)^{-5} - \frac{(100000 - B)(0.05)}{0.05^2} + 800,000(1.05)^8(1.05)^{-9}$$

So setting $P'(0.05) = 0$ and solving for B gives $B = 76470.58$ as the amount to be invested in bonds with the remainder in stock. With this allocation, one finds $P''(.05) = 20(76470.58)(1.05)^{-2} + 2(100000 - 7647.58)(.05)^{-2} - 720,000(1.05)^{-9} = 965786.7413 > 0$, so the portfolio is optimal ■

Example 55.5
Find the duration and convexity of a 20 year zero coupon bond assuming that the interest rate is 7% effective.

Solution.
The duration is 20 since the par value is repaid at the end of 20 years and this is the only payment made by the bond. Now, per dollar of par value, $P(i) = (1+i)^{-20}$, so that the convexity

$$\frac{P''(i)}{P(i)} = \frac{420}{(1+i)^2}.$$

Substituting $i = 0.07$ gives the convexity as 366.844 ■

Convexity of a Portfolio
Consider a portfolio consisting of n bonds. Let bond k have current price $P_k(i)$ and convexity $\overline{c}_k(i)$. Then the current value of the portfolio is

$$P(i) = P_1(i) + P_2(i) + \cdots + P_n(i).$$

Let $\overline{c}$ denote the convexity of the portfolio. Then, we have

$$\begin{aligned}
\overline{c} =& \frac{P''(i)}{P(i)} = \frac{P_1''(i)}{P(i)} + \frac{P_2''(i)}{P(i)} + \cdots + \frac{P_n''(i)}{P(i)}\\
=& \frac{P_1(i)}{P(i)}\left(\frac{P_1''(i)}{P_1(i)}\right) + \frac{P_2(i)}{P(i)}\left(\frac{P_2''(i)}{P_2(i)}\right) + \cdots + \frac{P_n(i)}{P(i)}\left(\frac{P_n''(i)}{P_n(i)}\right)\\
=& \frac{P_1(i)}{P(i)}\overline{c}_1 + \frac{P_2(i)}{P(i)}\overline{c}_2 + \cdots + \frac{P_n(i)}{P(i)}\overline{c}_n
\end{aligned}$$

Thus, the convexity of a portfolio can be calculated as the weighted average of the bonds' convexities, using the market values of the bonds as the weights.

Practice Problems

Problem 55.1 ‡

Which of the following statements about immunization strategies are true?
I. To achieve immunization, the convexity of the assets must equal the convexity of the liabilities.
II. The full immunization technique is designed to work for any change in the interest rate.
III. The theory of immunization was developed to protect against adverse effects created by changes in interest rates.

Problem 55.2

Nova Inc. has liabilities of 10 due in 1, 4, and 7 years. What asset incomes must the company arrange for in years 1 and 6 to immunize their cash flow, assuming an annual interest rate of 10% on all transactions?

Problem 55.3

A bank is required to pay 1,100 in one year. There are two investment options available with respect to how money can be invested now in order to provide for the 1,100 payback:
(i) a non-interest bearing cash fund, for which x will be invested, and
(ii) a two-year zero-coupon bond earning 10% per year, for which y will be invested.
Based on immunization theory, develop an asset portfolio that will minimize the risk that liability cash flows will exceed asset cash flows. Assume the effective rate of interest is equal to 10% in all calculations.

Problem 55.4

A \$5,000 payment is owed from Abby to Ben two years from now. Abby wants to set up an investment fund to meet this obligation, but the only investments she has available are a money market fund (currently earning 8%, but the rate changes daily), and a five-year zero-coupon bond earning 8%. Use an immunization framework to determine the amount of money Abby should invest now in each of the two investment vehicles. Assume an effective annual interest rate of 8% for present value calculations.

Problem 55.5

For the assets in Problem 55.4, find the modified duration and convexity.

Problem 55.6

Find the convexity of the following investments,assuming the effective rate of interest is 8%:
(a) A money market fund.
(b) A 10-year zero coupon bond.
(c) A preferred stock paying level dividends into perpetuity.

Problem 55.7
Find the convexity of a loan repaid with equal installments over n periods if $i = 0$.

Problem 55.8
A common stock pays dividends at the end of each year. It is assumed that each dividend is 4% greater than the prior dividend and the effective rate of interest is 8%. Find the convexity of this common stock.

Problem 55.9
Derive the following relationship between modified duration and convexity:

$$\frac{d}{di}\overline{\nu} = \overline{\nu}^2 - \overline{c}.$$

Problem 55.10
Suppose a company must make payments of 300 at time $t = 3$ and 500 at time $t = 5$, and the annual interest rate is 4%. Find asset income at times $t = 0$ and $t = 6$ so that the portfolio will be immunized.

56 Full Immunization and Dedication

In this section we discuss the two other strategies of immunization: Full immunization and dedication.

Full Immunization
Full immunization is an extension to the Redington immunization discussed in Section 50. While Redington immunization works for small changes in i, full immunization can be applied for all changes of i. We will say that a portfolio is **fully immunized** if $PV_A(i+\epsilon) > PV_L(i+\epsilon)$ for all $\epsilon > 0$ or equivalently $PV_A(i) \geq PV_L(i)$ for all positive i.
Suppose we have one liability outflow L_k at time k. The concept is to hold two assets, one that will produce a cash inflow, A, prior to the liability outflow (say at time $k-a$) , and another that will produce a cash inflow subsequent to the liability outflow (say at time $k+b$). We use the force of interest δ that is equivalent to i. The **full immunization conditions** for this single liability cash flow are:
(1) Present value of assets = Present value of liability.
(2) Modified duration of assets = Modified duration of liability.
(3) The asset cash flows occur before and after the liability cash flow.

Example 56.1
An insurance company has committed to make a payment of \$100,000 in 10 years. In order to fund this liability, the company has invested \$27,919.74 in a 5-year zero-coupon bond and \$27,919.74 in a 15-year zero-coupon bond. The annual effective yield on all assets and liabilities is 6%. Determine whether the company's position is fully immunized.

Solution.
Condition 1: Present value of assets = Present value of liability.
We have $PV_L = 100,000(1.06)^{-10} = 55,839.48$ and $PV_A = 27,919.74 \times 2 = 55839.48$, so condition (1) is met.

Condition 2: Modified duration of assets = Modified duration of liability.
We have

$$\text{Modified duration of liabilities} = \frac{10(100{,}000)(1.06)^{-10}}{100{,}000(1.06)^{-10}} = 10$$

and

$$\text{Modified duration of assets} = \frac{5(27{,}919.74)+15(27{,}919.74)}{55{,}839.48} = 10.$$

Thus, condition (2) is true.

Condition 3: The asset cash flows occur before and after the liability cash flow. Thus, the position is fully immunized ■

Now, conditions (1) and (2) leads to the following system

$$\begin{aligned} P(\delta) =& Ae^{-(k-a)\delta} + Be^{-(k+b)\delta} - L_k e^{-k\delta} = 0 \\ P'(\delta) =& -A(k-a)e^{-(k-a)\delta} - B(k+b)e^{-(k+b)\delta} + kL_k e^{-k\delta} = 0. \end{aligned}$$

The first condition implies

$$L_k = Ae^{a\delta} + Be^{-b\delta}.$$

The second condition implies

$$\begin{aligned} -A(k-a)e^{-(k-a)\delta} - B(k+b)e^{-(k+b)\delta} + kL_k e^{-k\delta} =& 0 \\ -Ake^{-(k-a)\delta} + Aae^{-(k-a)\delta} - Bke^{-(k+b)\delta} - Bbe^{-(k+b)\delta} + kL_k e^{-k\delta} =& 0 \\ -k\left[Ae^{-(k-a)\delta} + Be^{-(k+b)\delta} - L_k e^{-k\delta}\right] + \left[Aae^{-(k-a)\delta} - Bbe^{-(k+b)\delta}\right] =& 0 \\ Aae^{-(k-a)\delta} - Bbe^{-(k+b)\delta} =& 0 \\ Aae^{a\delta} - Bbe^{-b\delta} =& 0 \end{aligned}$$

Thus, we obtain the system

$$\begin{aligned} Ae^{a\delta} + Be^{-b\delta} =& L_k \\ Aae^{a\delta} =& Bbe^{-b\delta} \end{aligned}$$

There are exceptional cases when the system has no solutions, but if the known quantities are (i) a, b (ii) B, b (iii) A, a or (iv) A, b then unique values of the other two quantities can be found. Solving the last equation for B we find

$$B = A\left(\frac{a}{b}\right)e^{(a+b)\delta}.$$

We next show that conditions (1)-(3) achieve the required immunization. That is, we will show that for any $\delta' \neq \delta$ there is a surplus, i.e. $PV_A(\delta') > PV_L(\delta')$.
Let $S(\delta')$ denote the present value, at force of interest δ' per annum, of the total-assets less the present value of the liability, i.e. the surplus function

$$S(\delta') = PV_A(\delta') - PV_L(\delta') = e^{-k\delta'}(Ae^{a\delta'} + Be^{-b\delta'} - L_k).$$

Then we have

$$\begin{aligned}S(\delta') =& e^{-k\delta'}[Ae^{a\delta'} + Be^{-b\delta'} - L_k]\\ =& e^{-k\delta'}[Ae^{a\delta'} + Be^{-b\delta'} - (Ae^{a\delta} + Be^{-b\delta})]\\ =& e^{-k\delta'}Ae^{ab}\left[e^{a(\delta'-\delta)} + \frac{a}{b}e^{-b(\delta'-\delta)} - \left(1 + \frac{a}{b}\right)\right]\end{aligned}$$

Next, consider the function

$$f(x) = e^{ax} + \frac{a}{b}e^{-bx} - \left(1 + \frac{a}{b}\right).$$

Clearly, $f(0) = 0$. Moreover, since $f'(x) = a(e^{ax} - e^{-bx})$, one can easily check the following

$$\begin{aligned}f'(0) =& 0\\ f'(x) >& 0 \; for \; x > 0\\ f'(x) <& 0 \; for \; x < 0\end{aligned}$$

Thus, $f(x) > 0$ for all $x \neq 0$. This shows that $S(\delta') > 0$ for all $\delta' \neq \delta$ as required.
For a portfolio with multiple liabilities, repeat the above process for each liability outflow. There will be two asset inflows for each liability outflow.
Full immunization is like Redington immunization in the sense that portfolio must be rebalanced periodically so that to ensure that modified duration of assets is equal to modified duration of liabilities.

Example 56.2
An investor has a single liability of \$1000 due in 15 years' time. The yield on zero coupon bonds of any term is currently 4% per annum, and the investor possesses cash equal to the present value of his liability, i.e. $1000(1.04)^{-15} = 555.26$. He wishes to invest in 10-year and 20-year zero-coupon bonds in such a way that he will make a profit on any immediate change in the force of interest. How much of each security should he buy, and how large a profit will he make if the rate of interest per annum immediately becomes 0.01, 0.02, 0.03, 0.05, 0.06, 0.07,or 0.08?

Solution.
Letting $\delta = \ln 1.04$ we find

$$\begin{aligned}Ae^{5\delta} + Be^{-5\delta} =& 1000\\ 5Ae^{5\delta} - 5Be^{-5\delta} =& 0\end{aligned}$$

or equivalently, we have the system of two equations

$$A(1.04)^5 + B(1.04)^{-5} = 1000$$
$$A(1.04)^5 - B(1.04)^{-5} = 0.$$

These equations may easily be solved, giving $A = 410.96$ and $B = 608.32$. The quantities of zero-coupon bonds which he should buy are therefore those providing \$410.96 at time 10 years and \$608.32 at time 20 years. The profit on an immediate change in the rate of interest to i per annum is,

$$S(i) = (1+i)^{-15}[410.96(1+i)^5 + 608.32(1+i)^{-5} - 1000].$$

In the following table we give the values of the liability (\$1000$(1+i)^{-15}$), the assets ($410.96(1+i)^{-10} + 608.32(1+i)^{-20}$) and the profit to the investor for each of the specified rates.

Annual interest rate	PV_L	PV_A	Present value of profit
0.00	1000.00	1019.28	19.28
0.01	861.35	870.58	9.23
0.02	743.01	746.51	3.50
0.03	641.86	642.61	0.75
0.04	555.26	555.26	0.00
0.05	481.02	481.56	0.54
0.06	417.27	419.16	1.89
0.07	362.45	366.11	3.66
0.08	315.24	320.87	5.63

Dedication

In this section we discuss an approach for matching liabilities and assets. This approach is called **dedication** or **absolute matching.** In this approach, an company structures an asset portfolio in such a fashion that the cash inflow that will be generated from assets will exactly match the cash outflow from the liabilities.

If the above strategy is achieved, the company has full protection against any movement in interest rates. However, the following problems exist with implementing this strategy.

1. Cash flows are usually not that predictable on either the asset or the liability side.
2. If the liabilities are long term in nature, it might be impossible to find assets to exactly match the liabilities without creating reinvestment risk.
3. The yield rate on a fund structured with the major restrictions imposed by absolute matching may be less than that on a fund for which more flexibility is available, and the extra return may overshadow the advantages of absolute matching in importance.

Example 56.3
An insurance company accepts an obligation to pay 10,000 at the end of each year for 2 years. The insurance company purchases a combination of the following two bonds (both with \$1,000 par and redemption values) in order to exactly match its obligation:
Bond A: A 1-year 5% annual coupon bond with a yield rate of 6%.
Bond B: A 2-year 8% annual coupon bond with a yield rate of 7%.
(a) Find the numbers (which need not be integers) of each bond the insurer must purchase to exactly match its obligations.
(b) Find the total cost to the insurer of purchasing the needed numbers of bonds.

Solution.
(a) We have the following chart

Period	Bond A	Bond B	Liabilities
1	1050	80	10,000
2		1080	10,000

Let a be the number of bonds A purchased and b that of bonds B. Then we have the following system of two equations in two unknowns

$$\begin{aligned} 1050a + 80b =& 10,000 \\ 1080b =& 10,000 \end{aligned}$$

Solving for a and b we find $a = 8.8183$ bonds and $b = 9.2593$ bonds.
(b) The answer is

$$8.8183(1050\nu_{0.06}) + 9.2593(80\nu_{0.07} + 1080\nu_{0.07}^2) = \$18161.82 \blacksquare$$

Practice Problems

Problem 56.1
An insurance company has an obligation to pay \$1,000,000 at the end of 10 years. It has a zero-coupon bond that matures for \$413,947.55 in 5 years, and it has a zero-coupon bond that matures for \$864,580.82 in 20 years. The effective yield for assets and liability is 10%.
(a) Determine whether the company's position is fully immunized.
(b) What is the level of surplus if the interest rate falls to 0%?
(c) What is the level of surplus if the interest rate rises to 80%?

Problem 56.2 ‡
A company must pay liabilities of 1000 and 2000 at the end of years 1 and 2, respectively. The only investments available to the company are the following two zero-coupon bonds:

Maturity(years)	Effective annual yield	Par
1	10%	1000
2	12%	1000

Determine the cost to the company today to match its liabilities exactly.

Problem 56.3 ‡
An insurance company accepts an obligation to pay 10,000 at the end of each year for 2 years. The insurance company purchases a combination of the following two bonds at a total cost of X in order to exactly match its obligation:
(i) 1-year 4% annual coupon bond with a yield rate of 5%
(ii) 2-year 6% annual coupon bond with a yield rate of 5%.
Calculate X.

The following information applies to Problems 56.4 - 56.6

Joe must pay liabilities of 1,000 due 6 months from now and another 1,000 due one year from now. There are two available investments:
• a 6-month bond with face amount of 1,000, a 8% nominal annual coupon rate convertible semi-annually, and a 6% nominal annual yield rate convertible semiannually; and
• a one-year bond with face amount of 1,000, a 5% nominal annual coupon rate convertible semi-annually, and a 7% nominal annual yield rate convertible semiannually.

Problem 56.4 ‡
How much of each bond should Joe purchase in order to exactly (absolutely) match the liabilities?

Problem 56.5 ‡
What is Joe's total cost of purchasing the bonds required to exactly (absolutely) match the liabilities?

Problem 56.6 ‡
What is the annual effective yield rate for investment in the bonds required to exactly (absolutely) match the liabilities?

Problem 56.7
John wants to absolutely match a debt that he owes under which he must make payment of 1000 in one year and 2000 in two years. He can purchase the following bonds:
(1) Bond A is a two year bond with annual coupons of 100 and a maturity value of 1000.
(2) Bond B matures in one year for 1000 and pays a coupon of 80.
Calculate the amount of Bond B that John should purchase.

An Introduction to the Mathematics of Financial Derivatives

What is a financial derivative? A **financial derivative** is a financial contract (between two parties) that *derives* its value from the value of some underlying asset. For example, a homeowner's insurance policy promises that in the event of a damage to your house, the insurance company will compensate you for at least part of the damage. The greater the damage, the more the insurance company will pay. Your insurance policy thus derives its value from the value of your house and therefore is a derivative.

In this chapter, we introduce four financial derivatives: forward contracts, future contracts, options (call and put) and swaps and we discuss some of the related topics.

57 Financial Derivatives and Related Issues

In this section we introduce the concept of financial derivatives and discuss some of the relevant topics of derivatives.

What is a Financial Derivative?
A **financial derivative** is a contract (between two parties) that *derives* its value from the value of some underlying asset(s). Examples of underlying assets include financial assets (such as stocks or bonds), commodities (such as gold or oil), market indices (such as S&P 500 or FTSE 100), interest rates (LIBOR rates) and many others.
An example of a derivative is the following: Party A makes an agreement with Party B regarding the price of gold. If the price of one ounce of gold in one year is above \$300, Party A agrees to pay Party B the amount of \$1. Otherwise, Party B will pay Party A \$1. The agreement is a derivative since the outcome depends on the price of gold. Note that this agreement is a kind of a "bet" on the price moves of gold. In general derivatives can be viewed as bets on the price of something.
The most common types of financial derivatives that we will discuss in this book are futures, forwards, options, and swaps.

Derivatives Trading Markets
Derivatives can be traded through organized exchanges such as The Chicago Board of Trade(CBOT), the New York Stock Exchange (NYSE), the New York Mercantile Exhange (NYMEX) and many others, or through contracts negotiated privately between two parties, referred to as over−the−counter (abbreviated OTC). Financial markets are usually regulated by the Securities and Exchange Commission (SEC) and the Commodity Futures Trading Commission (CFTC).
Financial markets serve six basic functions. These functions are briefly listed below:
• Borrowing and Lending: Financial markets permit the transfer of funds (purchasing power) from one agent to another for either investment or consumption purposes.
• Price Determination: Financial markets provide vehicles by which prices are set both for newly issued financial assets and for the existing stock of financial assets.
• Information Aggregation and Coordination: Financial markets act as collectors and aggregators of information about financial asset values and the flow of funds from lenders to borrowers.
• Risk Sharing: Financial markets allow a transfer of risk from those who undertake investments to those who provide funds for those investments.
• Liquidity: Financial markets provide the holders of financial assets with a chance to resell or liquidate these assets.
• Efficiency: Financial markets reduce transaction costs and information costs.

Most Common Uses of Derivatives

Below are some of the motives for someone to use derivatives:
(1) to reduce (or **hedge**) exposure to risk. For example, a wheat farmer and a wheat miller could enter into a futures contract to exchange cash for wheat in the future. Both parties have reduced a future risk: for the wheat farmer, the uncertainty of the price, and for the wheat miller, the availability of wheat.
(2) to speculate expected changes in future prices with the hope of making profit. In this case, speculation increases the exposure to risk. The potential gain or loss can be **leveraged** (i.e. magnified) relative to the initial investment.
(3) to reduce transaction costs, such as commissions and other trading costs.
(4) to maximize return on investments through asset management activities, tax loopholes, and regulatory restrictions. For example, a company can use derivatives to produce temporary losses to lower its taxes. We refer to this motive as **regulatory arbitrage.**

Example 57.1

Weather derivatives are financial instruments that can be used by organizations or individuals as part of a risk management strategy to reduce risk associated with adverse or unexpected weather conditions (rain/temperature/snow). Two types of weather derivatives are heating degree-day (HDD) and cooling degree-day (CDD) future contracts. These future contracts make payments based on whether the temperature is abnormally hot or cold.
(a) Explain why a soft-drink manufacturer might be interested in such a contract.
(b) The business buys these future contracts to hedge against temperature-related risk. Who is the other party accepting the risk?

Solution.

(a) Soft drink sales greatly depend on weather. Generally, warm weather boosts soft drink sales and cold weather reduces sales. A soft drink producer can use weather futures contracts to reduce the revenue swing caused by weather and smooth its earnings. Shareholders of a company generally want the earnings to be steady. They don't want the management to use weather as an excuse for poor earnings or wild fluctuations of earnings.
(b) Anyone who can predict a weather index can enter into weather futures and make a profit. However, since predicting weather is not usually easy and accurate a loss may occur ■

Perspectives on Derivatives

There are three different user perspectives on derivatives:
• **The end-user perspective.** End-users include corporations, investment managers, and investors. End-users use derivatives in order to achieve a specific goal or goals such as managing risk, speculating, reducing costs, or avoiding regulations.

• **The market-maker perspective.** These are traders or intermediaries between different end-users. They buy from end-users who want to sell (usually at a low price) and sell to end-users who want to buy (usually at a higher price.) Also, market-users might charge commissions for trading transactions.
• **The economic observer** such as a regulator or a research economist, whose role is to watch and even sometimes regulate the markets.

Traders of Derivatives
There are three main traders of derivatives: hedgers, speculators and arbitrageurs.
• **Hedgers** use derivatives to reduce risk that they face from potential future movements in market variables.

Example 57.2
Suppose that it is August 16 2007 and, firm A, a company based in the US knows that it will pay £10 million on November 16, 2007 for goods it has purchased from a British firm. Firm A can buy pounds from a bank in a three-month forward contract and thus hedge against foreign exchange risk. Suppose the agreement is to buy £1 for \$1.55.
(a) What would the firm A pay on November 16, 2007?
(b) Suppose that on November 16, 2007 the exchange rate is \$3 for £1. What would firm A pay on Novemebr 16, 2007 in the absence of hedging?

Solution.
(a) The firm has hedged against exchange rate volatility. The amount to be paid in US dollars on November 16, 2007 is $1.55 \times 10,000,000 = \$15.5$ millions.
(b) The firm would have to pay $3 \times 10,000,000 = \$30$ millions ■

• **Speculators** use derivatives to bet on the future direction of a market variable. They can, for example, buy a put option on a stock if they think it will go down. If they think that the price of the stock will go up they will buy a call option. Another example of a speculator is an investor in weather derivatives who buys temperature-related future contracts.

• **Arbitrageurs** take offsetting positions in two or more instruments to lock in a riskless profit if securities are inconsistently priced.

Example 57.3
Suppose that firm A is dually listed on both the NYSE and LSE. On the NYSE its share is trading at \$152 and on the LSE its share is trading at £100. The exchange rate is \$1.55 per pound. An arbitrageur makes a riskless profit by simultaneously buying 100 shares on the NYSE and sell them on the LSE. What is the amount of this profit?

Solution.
He buys the shares on the NYSE for \$152 a share. He sells the shares on the LSE for \$155 a share. So the profit per share is \$3. Total profit from the 100 shares is then \$300 ■

Practice Problems

Problem 57.1
Which of the following is a derivative?
(I) A bushel of corn
(II) A contract to sell 100 bushels of corn
(III) Short sale of a stock.

Problem 57.2
The Security and Exchange Commission (SEC) holds primary responsibility for enforcing the federal securities laws and regulating the securities industry. Which of the following describes the perspective of SEC on derivatives?
(I) The end-user perspective
(II) The market-maker perspective
(III) The economic observer.

Problem 57.3
List four motives for using derivatives.

Problem 57.4
Company A based in the US buys goods from a British company and is required to pay in British pounds for the cost of the goods. Company A buys a derivative that allows it to use the current exhange rate after six months from today. Which of the following characterizes company A uses of the derivative?
(I) Hedging
(II) Speculation
(III) Reduced transaction costs
(IV) Regulatory loop holes.

Problem 57.5
You buy 100 shares of a stock through a brokerage firm. Which of the following describes your perspective of using derivatives?
(I) The end-user perspective
(II) The market-maker perspective
(III) The economic observer.

Problem 57.6
You buy 100 shares of a stock through a brokerage firm. Which of the following describes brokerage firm perspective of using derivatives?

(I) The end-user perspective
(II) The market-maker perspective
(III) The economic observer.

Problem 57.7
You work for a firm who has a reimbursement program for college credits. The program reimburses 100% of costs for an "A", 75% of costs for a "B", 50% for a "C" and 0% for anything less. Would this program be considered an example of a derivative? Explain.

Problem 57.8
Explain why the following firms might be interested in buying HDD or DCC future contracts.
(a) Ski-resort owner.
(b) Electric utilities.
(c) Amusement park owner.

Problem 57.9
Match each of the following market traders: (a) hedgers (b) speculators (c) arbitrageurs with one of the following roles:
(I) profit from perceived superior expectations
(II) trade to eliminate or reduce future price risk
(III) attempt to profit when the same security or commodity is trading at different prices in two or more markets.

Problem 57.10
Explain carefully the difference between (a) hedging (b) speculation and (c) arbitrage.

58 Derivatives Markets and Risk Sharing

As pointed out in the previous section, one of the basic function of derivatives markets is risk sharing. Risk is a major feature of economic activity. In this section we discuss the role of financial markets as a vehicle of sharing risks.
Risk-sharing is one of the most important functions of financial markets. To illustrate this concept, consider auto insurance companies. These companies collect premiums for auto insurance policies. The total premiums collected are then being available to help those who get involved into car wrecks. Thus, those policyholders who had no wrecks in their records have basically lost their premiums. However, their premiums went to help those who needed it.
A similar scenario occurs in the business world. Some companies profit and others suffer. Thus, it makes sense to have a mechanism enabling companies to exchange financial risks. Share risking mechanisms should benefit everyone.
Another example of risk sharing is the use of the so-called **catastrophe bonds**. For example, an insurance company that provides hurricane insurance for Florida residents will usually face large insurance claims in the case of a devastating hurricane happening. For that reason, the insurance company usually either issues or buys from reinsurers cat bonds enabling the company to share risks with the bondholders. Catastrophe bondholders receive interest rates at levels commensurate with the risk that they may lose all of their principal on the occurrence of a major hurricane.
The sponsor issues cat bonds and typically invests the proceeds from the bond issuance in low-risk securities. The earnings on these low-risk securities, as well as insurance premiums paid to the sponsor, are used to make periodic, variable rate interest payments to investors.
Sponsors of cat bonds include insurers, reinsurers, corporations, and government agencies.

Diversifiable Risk Versus Non-Diversifiable Risk
Risk management is one of the topics discussed in portfolio theory. The risk level of an asset can be categorized into two groups: the diversifiable risk and the non-diversifiable risk. Before we discuss the difference between diversifiable and non-diversifiable risks, we need to first understand the term **diversification.** The act of diversification implies that an individual (or a firm) allocates his/her wealth among several types of investments rather than just one investment. In other words, diversification means spreading the risk of a portfolio by investing in several different types of investments rather than putting all the money in one investment.
Diversifiable risk (also known as **non-systematic** risk) is a risk that can be reduced or eliminated by combining several diverse investments in a portfolio.

Example 58.1
Suppose you are an avid Apple computer user, and you decide to invest all your money in Apple stocks. Are you making a sound decision? If not, then what is a good investment strategy?

Solution.
The answer is no. If you have done so, you will be very vulnerable in the last few months when Apple announced losses and a major change in management. However, you can reduce the risk of your portfolio if you invest in other computer companies such IBM, Compaq, etc. However, this is not a very well diversified portfolio because you have invested your money in the computer industry. As a result, you will still be vulnerable to changes in the computer industry. It is wise to have a portfolio consisting of shares in firms from various industries such as real estate, electronics, biomedical and so on ■

Non-diversifiable risk is the part of an asset's risk that cannot be eliminated through diversification. This type of risk is also known as **market risk** or **systematic risk**. Examples of non-diversifiable risks are natural disasters, wars, and economic factors such as inflation or unemployment levels events that can influence the entire market and usually affects your entire portfolio.

Example 58.2
Which of the following can be categorized as a non-diversifiable risk?
(a) Recession.
(b) The fall of a stock share.
(c) Government defaults in paying bonds.

Solution.
Obviously only (a) and (c) might have major impact of one's entire portfolio ■

Practice Problems

Problem 58.1
Explain why a health insurance policy is a risk sharing derivative.

Problem 58.2
List the two types of risk involved in a portfolio.

Problem 58.3
The act of diversification implies that an individual (or a firm) allocates his/her wealth
(I) among several types of assets
(II) in just one asset

Problem 58.4
Determine whether each of the following events is considered a diversifiable or non-diversifiable risk for a portfolio:
(a) A software firm's best programmer quits.
(b) Oil-prices on international markets suddenly increase.
(c) Congress votes for a massive tax cut.
(d) A low-cost foreign competitor unexpectedly enters a firm's market.

Problem 58.5
Determine whether each of the following events is considered a diversifiable or non-diversifiable risk for a portfolio:
(a) A wildcat strike is declared.
(b) Oil is discovered on a firm's property.
(c) The Federal Reserve institutes a restrictive monetary policy.
(d) Long term interest rates rise sharply.

Problem 58.6
An earthquake linked bond is called
(a) callable bond
(b) perpetual bond
(c) cat bond
(d) serial bond

Problem 58.7
Decide whether the risk is a diversifiable risk or non-diversifiable risk:

(a) Management risk
(b) Inflation risk
(c) Sociopolitical risk
(d) Credit risk
(e) Currency risk
(f) Interest rate risk
(g) Liquidity risk.

Problem 58.8
True or false: Cat bonds are used by sponsors as a hedge against natural disasters.

Problem 58.9
Which of the following is not a term that refers to non-systematic risk?
(A) Non-diversifiable risk
(B) Market risk
(C) Diversifiable risk
(D) Both (A) and (B)

Problem 58.10
In 9/11, immediately after the terrorist attacks the stock market fell significantly. This is an example of
(A) Systematic risk
(B) Market risk (C) Non-diversifiable risk (D) All of the above

Problem 58.11
Recently in California the workers at Kroger supermarkets had been on strike. This strike would most likely have been seen as an example of
(A) Non-systematic risk
(B) Non-diversifiable risk
(C) Diversifiable risk
(D) None of the above

Problem 58.12
Explain how cat bonds work.

59 Forward and Futures Contracts: Payoff and Profit Diagrams

In this section we introduce two examples of derivatives: forward and futures contracts. We also analyze the possible payoffs and profits of these two securities.
Forwards and futures are very similar as contracts but differ in pricing and trading mechanisms. For example, forwards are traded over the counter (i.e. contract specification can be customized) whereas futures are traded on exchange markets (where contracts are standardized). For the purpose of this section, we think of them as interchangeable. A more detailed discussion of futures will be covered in Section 72.
A **forward contract** (or a **futures contract**) is a commitment to purchase at a future date, known as the **expiration date**, the **delivery date** or the **maturity date**, a given amount of a commodity or an asset at a price agreed on today. The price fixed now for future exchange is called the **forward price** or the **delivery price**. The asset or commodity on which the forward contract is based is called the **underlying asset**. Apart from commissions and bid-ask spreads, a forward contract requires no initial payment or premium between the two parties. At the delivery date, cash is exchanged for the asset. Examples of forward contracts include
• A **bond forward** is an obligation to buy or sell a bond at a predetermined price and time.
• An **equity forward** is an obligation to buy or sell an equity at a predetermined price and time.
• A **gold forward** is an obligation to deliver a specified quantity of gold on a fixed date and receive a fixed delivery price.
• A **foreign exchange forward** is an obligation to buy or sell a currency on a future date for a predetermined fixed exchange rate. This type of forwards is used by firms as a protection against fluctuation in foreign currency exchange rates.
• An **interest rate forward** is used to lock-in future interest rates. For example, a firm who is planning for a loan in six months can buy forward contract that lock-in the rate at the present. This rate will be the rate that will apply when the loan is exercised in six months. The risk here is that the firm will stuck with borrowing money at the stated rate so any decline of interest rates before the six months period will cause losses for the firm.

Example 59.1
Explain in words the meaning of the following: A 3-month forward contract for 1,000 tons of soybean at a forward price of \$165/ton.

Solution.
In this contract, the buyer is committed to buy 1000 tons of soybean from the seller in three months for a price of \$165 a ton ■

Payoff on a Forward Contract

Forward contracts are privately executed between two parties. The buyer of the underlying commodity or asset is referred to as the **long side** whereas the seller is the **short side.** The obligation to buy the asset at the agreed price on the specified future date is referred to as the **long position.** A long position profits when prices rise. The obligation to sell the asset at the agreed price on the specified future date is referred to as the **short position.** A short position profits when prices go down.

What is the **payoff** of a forward contract on the delivery date? Let T denote the expiration date, K denote the forward price, and P_T denote the spot price (or market price) at the delivery date. Then

• for the long position: the payoff of a forward contract on the delivery date is $P_T - K$;

• for the short position: the payoff of a forward contract on the delivery date is $K - P_T$.

Figure 59.1 shows a payoff diagram on a contract forward. Note that both the long and short forward payoff positions break even when the spot price is equal to the forward price. Also note that a long forward's maximum loss is the forward price whereas the maximum gain is unlimited. For a short forward, the maximum gain is the forward price and the maximum loss is unlimited.

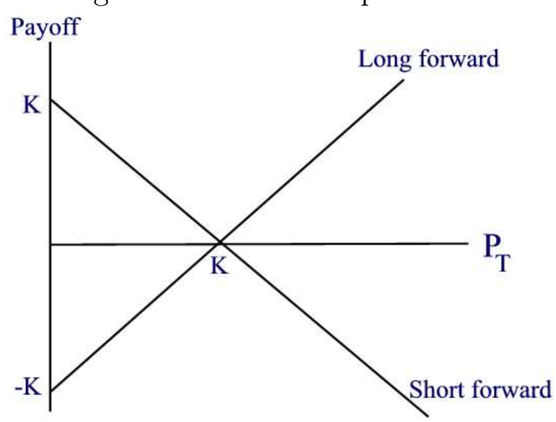


Figure 59.1

Payoff diagrams show the payoff of a position at expiration. These payoffs do not include any costs or gains earned when purchasing the assets today. Payoff diagrams are widely used because they summarize the risk of the position at a glance. We have pointed out earlier that the long makes money when the price rises and the short makes money when the price falls.

Example 59.2

An investor sells 20 million yen forward at a forward price of $0.0090 per yen. At expiration, the spot price is $0.0083 per yen.

(a) What is the long position payoff?
(b) What is the short position payoff?

Solution.
(a) At the expiration date, the long position's payoff is $(0.0083 - 0.009) \times 20 \cdot 10^6 = -\$14,000$, a loss of \$14,000
(b) The short position's payoff is \$14,000 that is a profit of \$14,000 ■

Example 59.3
Consider a forward contract on a stock with spot price of \$25 and maturity date of 3 months from now. The forward price is \$25.375. Draw the payoff diagram for both the long and short forward positions after 3 months.

Solution.
The diagram is given in Figure 59.2.

Figure 59.2

Note that both the long and short forward payoff positions break even when the price of the stock at maturity is equal to the forward price, i.e., at \$25.375 ■

Example 59.4
Consider the forward contract of the previous example. Suppose the investor decides to purchase the stock outright today for \$25. Is there any advantage of using the forward contract to buy the stock as opposed to buying it outright?
(a) Draw the diagram for the payoff to the long physical position as well as the payoff diagram to the long forward.
(b) If the spot price is \$0 in three months then find the long forward position's payoff.

(c) If the spot price of the stock in three months is \$25.372 then find the long physical position's payoff and the long forward's payoff.
(d) Describe the investement for the outright purchase as well as the long forward in order to own the stock.

Solution.
(a) The diagram is shown in Figure 59.3. Note that the value of the payoff of the physical position is equal to the spot price of the stock. The payoff diagram does not include costs when the stock is purchased outright.

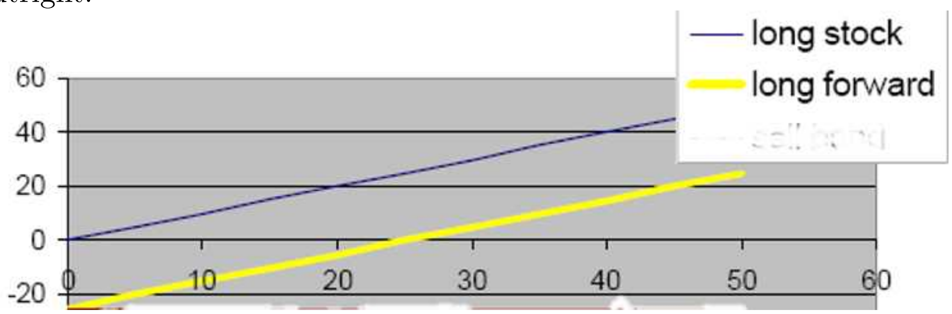


Figure 59.3

(b) The long forward's payoff is $0 - 25.372 = -\$25.372$.
(c) The payoff of the long physical position is \$25.372. The payoff of the long forward is $25.372 - 25.372 = \$0$.
(d) With the outright purchase, the investor invests \$25 now and owns the stock. With the forward contract, the investor invests \$0 now and \$25.372 after three months and owns the stock ■

Note that the payoff of the outright purchase in Figure 59.3 tells us how much money we end up with after 3 months, but does not account for the initial \$25 investment for purchasing the stock outright. Thus, from the graph one can not tell whether there is an advantage to either a forward purchase or an outright purchase. In order to have a fair comparison, the initial investments for both must be the same and then account for interest rate earned over the 3 months. We illustrate this point in the next two examples.

Example 59.5
Suppose the investor invests \$25 in a zero-coupon bond with par value \$25.372 and maturity date in three months along with the forward contract of the previous example. Note that this combined position and the outright purchase position each initially costs \$25 at time 0. Suppose the bond yield rate is 5.952% payable quarterly. Show that the combined position mimics the effect of buying the stock outright.

Solution.
The investor pays \$25 now for the bond. After three months the zero-coupon bond return is \$25.372. He then uses the proceeds to pay for the forward price of \$25.372 and owns the stock. This alternative is equivalent to outright purchase of the stock today for \$25 ■

Example 59.6
Suppose the investor borrows \$25 to buy the physical stock which costs \$25. Interest rate on the loan is 5.952% compounded quarterly. In this case, this position and the long forward position each initially costs \$0 at time 0. Show that this act of borrowing mimics the effect of entering into a long forward contract.

Solution.
The investor uses the borrowed money to buy the stock today. After three months, the investor repays $\$25 \times 1.01488 = \$25.372$ for the borrowed money. Thus, the investor invested nothing initially, and after three months he paid \$25.372 and owned the stock. This shows that borrowing to buy the stock mimics the effect of entering into a long forward contract ■

In the above two examples, we conclude that the forward contract and the cash position are equivalent investments, differing only in the timing of the cash flows.
Besides payoff diagrams, one defines a profit diagram. Recall that a payoff diagram graphs the cash value of a position at maturity. A **profit diagram** subtracts from the payoff the future value of the initial investment in the position. Geometrically, the profit diagram is a vertical shift of the payoff diagram by the amount of the future value of the initial investment. Since forward contracts require no initial investment, the payoff and profit diagrams coincide.

Example 59.7
A stock is priced at \$50 and pays no dividends. The effective annual interest rate is 10%. Draw the payoff and profit diagrams for a one year long position of the stock.

Solution.
The payoff of a stock you buy long (and therefore you own) is just the stock market value. Let's assume you want to find the profit after one year. Your profit is the payoff (market price) minus the future value of the initial investment over one year. If we denote the stock price by S then your profit will be $S - 1.1(50) = S - 55$. Figure 59.4 shows both the payoff and profit diagrams. Notice that the profit is 0 when the stock price is \$55 ■

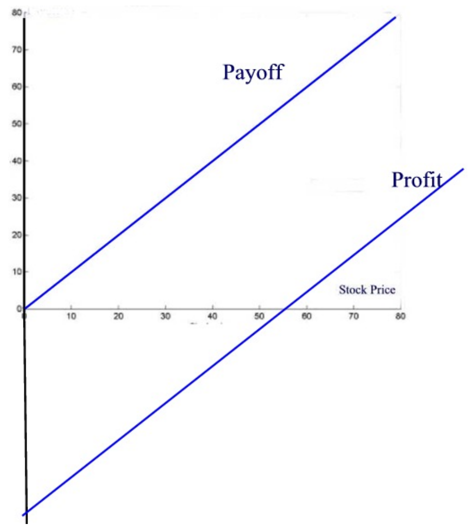


Figure 59.4

Example 59.8
Suppose we own an index which we want to be valued at \$1020 six months from today. For that we buy a forward contract with forward price of \$1020 and maturity date of six months. Consider investing \$1000 in a zero-coupon bond with six-month interest rate of 2%.
(a) Compare the payoff diagram of the combined position of the forward contract and the bond with the payoff diagram of the index after six months.
(b) Answer the same question for the profit diagrams.

Solution.
(a) The payoff of the forward contract and the bond in six months is

$$\text{forward} + \text{bond} = \text{spot price at expiration} - 1020 + (1000 + 1000(0.02)) = \text{spot price at expiration}$$

Thus, the payoff diagram of the combined position is the same as the payoff diagram of the index. Adding the bond shifts the payoff diagram of the forward vertically upward by \$1020.
(b) The profit diagram of the forward contract coincides with its payoff's diagram since there is no initial investment. The profit of the bond is the payoff minus the future value of the investment which comes out to 0. Thus, the profit diagram of the combined forward and bond is the same as the profit diagram of the forward ■

Cash-Settled Forward Contracts
A forward may be cash-settled, in which case the underlying asset and payment never exchange hands. Instead, the contract settles with a single payment in the amount spot price minus forward

price at delivery date. If the difference is positive, the short party pays the long party the difference. If it is negative, the long party pays the short party.

Example 59.9
Two parties agree today to exchange 500,000 barrels of crude oil for \$42.08 a barrel three months from today. Describe this forward when settled for cash
(a) if the spot price at the delivery date is \$47.36 a barrel;
(b) if the spot price at the delivery date is \$40.17 a barrel.

Solution.
If the forward were cash-settled then on the delivery date three months from today, no oil would change hands, and there would be no payment of $500,000 \times 42.08 = \$21.04$ millions.
(a) If the spot price at the delivery date was \$47.36 a barrel then the forward would settle with a single payment of

$$500,000(47.36 - 42.08) = \$2,640,000$$

made by the short party to the long party.
(b) If the spot price at the delivery date was \$40.17 a barrel then the forward would settle with a single payment of

$$500,000(42.08 - 40.17) = \$955,000$$

made by the long party to the short party ■

Credit Risk
We conclude this section by mentioning the credit risk (i.e. the risk that a party will not meet its contractual obligations) associated with forward contracts. Exchange-traded contracts typically require collateral in order to minimize this risk . In over-the-counter contracts, each party bears the other's credit risk.

Example 59.10
Creditwise, which is riskier a forward contract or a futures contract?

Solution.
Since forward contracts are traded on over-the-counter and futures contracts are traded on exchanges, forward contracts are more riskier than futures contracts ■

Practice Problems

Problem 59.1
What is the difference between a long forward position and a short forward position?

Problem 59.2
What is the difference between a long futures position and a short futures position?

Problem 59.3
A trader enters into a short forward contract on 100 million yen. The forward exchange rate is 189.4 yen per £. How much does the trader gain or lose if the exchange rate at the end of the contract is (a) 192.0 yen per £; (b) 183.0 yen per £?

Problem 59.4
A stock with no dividends has a current price of $50. The forward price for delivery in one year is $53. If there is no advantage of buying either the stock or the forward contract, what is the 1-year effective interest rate?

Problem 59.5
A default-free zero-coupon bond costs $91 and will pay $100 at delivery date in one year. What is the effective annual interest rate? What is the payoff diagram for the bond? What is the profit diagram for the bond?

Problem 59.6
Suppose that you enter into a long six-month forward position at a forward price of $50. What is the payoff in 6 months for prices of $40, $45, $50, $55, and $60?

Problem 59.7
Suppose that you enter into a short six-month forward position at a forward price of $50. What is the payoff in 6 months for prices of $40, $45, $50, $55, and $60?

Problem 59.8
Suppose shares of XYZ corporation are trading at $50 and currently pay no dividends. The forward price for delivery in 1 year is $55. Suppose the 1-year effective annual interest rate is 10%. Draw the payoff and profit diagrams of this forward contract for a long position.

Problem 59.9
A company agrees to buy 2,000 barrels of oil in two years; the current price is $63.50 per barrel. What is reasonable price to pay per barrel in two years?

Problem 59.10
Consider a forward contract of a stock index with maturity date six months from now and forward price of \$1020. Answer the following questions if this forward contract is settled for cash.
(a) Find the long and short positions payoffs if the spot price of the index at maturity is \$1040.
(b) What if the spot sprice is \$960?

Problem 59.11
You are a US exporter who will receive 5 millions Euro in six months time from the sale of your product in Spain. How can you hedge your foreign exchange risk ?

Problem 59.12
Consider a forward contract of a stock index with maturity date three months from now and forward price of \$930. Assume that the spot price of the index today is \$900. What is the profit or loss to a short position if the spot price of the market index rises to \$920 by the expiration date?

Problem 59.13
One ounce of gold today is being sold at the price of \$300 per ounce. An investor borrows \$300 with annual interest rate of 5% and buys one ounce of gold. The investor enters into a short forward contract to sell 1 oz. of gold in one year at \$340. What is the investor's profit in this transaction? What would be his profit if the original price of 1 oz. of gold is \$323.81 and the loan is \$323.81?

Problem 59.14
An investor enters into a short forward contract to sell 100,000 British pounds for US dollars at an exchange rate of 1.50 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.49 and (b) 1.52?

Problem 59.15
A forward contract is made for the delivery of a commodity 10 months from today at a forward price of \$200 per unit of the commodity.
(A) The purchaser of the forward contract pays the forward price, \$200, immediately but must wait for 10 months to receive the commodity.
(B) The holder of a forward contract always has the right to exchange it for an identical futures contract at any time before the delivery date.
(C) The seller of the forward contract has an obligation to deliver the commodity 10 months from today in return for the spot price on the delivery date.
(D) The seller of the forward contract has an obligation to deliver the commodity 10 months from today in return for \$200 per unit, payable on delivery.

Problem 59.16
The main difference between forward and futures contracts is:
(A) Futures contracts are traded every day (i.e., frequently) in organized exchanges prior to the delivery date specified in the contract.
(B) For a futures contract there is never any obligation to deliver, or to receive, the underlying asset.
(C) Forward contracts always have delivery dates further from the present than futures contracts.
(D) The underlying assets in forward contracts are always physical commodities (e.g., wheat, oil or silver), while for futures contracts the underlying assets could be anything (e.g. weather indices or notional bank deposits).

60 Call Options: Payoff and Profit Diagrams

One problem with a forward contract is the obligation of the buyer to pay the forward price at the expiration date which causes a loss if the spot price is below the forward price. So it is natural to wonder if there is a type of contract where the buyer has the option not to buy the underlying asset. The answer is affirmative thanks to call options.

An **option** is a contract to buy or sell a specific financial product such as a security or a commodity. The contract itself is very precise. It establishes a specific price, called the **strike price** or **exercise price** at which the contract may be exercised, or acted on, and it has an **expiration date.** When an option expires, it no longer has value and no longer exists.

There are two types of options: calls and puts. In this section we discuss call options and in the next section we discuss the put options.

In a **call** option the call buyer expects that the price of an asset may go up. The buyer pays an upfront premium that he will never get back. He has the right to exercise the option at or before the expiration date depending on the option's style(to be introduced below). The call seller, also known as the **option's writer**, receives the premium. If the buyer decides to exercise the option, then the seller has to sell the asset at the strike price. If the buyer does not exercise the option, then the seller profit is just the premium. Thus, a call option is a contract where the buyer has the right to buy but not the obligation to buy.

An **option style** governs the time at which exercise can occur. A **European call option** allows the holder to exercise the option (i.e., to buy) only on the option expiration date. An **American call option** allows exercise at any time during the life of the option. A **Bermuda call option** can be exercised at certain pre-specified dates before or at the expiration date. Unless otherwise specified, we will limit our discussions to European style.

For a call option, the buyer has a long position whereas the seller has a short position.

Example 60.1

Trader A (call buyer) purchases a call contract to buy 100 shares of XYZ Corp from Trader B (call seller) at \$50 a share. The current price is \$45 a share, and Trader A pays a premium of \$5 a share.

(a) If the share price of XYZ stock rises to \$60 a share right before expiration, then what profit does Trader A make if he exercised the call by buying 100 shares from Trader B and sell them in the stock market?

(b) If, however, the price of XYZ drops to \$40 a share below the strike price, then Trader A would not exercise the option. What will be his losses in this case?

Solution.

(a) Trader A total cost for owning the 100 shares before or on the expiration date is $50(100) + 5(100) = 5,500$. The revenue from selling the shares is $60(100) = 6,000$. Thus, Trader A makes a

profit of \$500 in this transaction.
(b) The losses will consist basically of the future value of premium paid, i.e. the future value of 5(100) = \$500 ■

Since a buyer of a call option will exercise the option only when the spot price is higher than the strike price, the buyer's payoff is then defined by the formula

$$\text{Buyer's call payoff} = \max\{0, \text{spot price at expiration} - \text{strike price}\}.$$

Example 60.2
Consider a call option with a strike price of \$500.
(a) What would be the payoff to the buyer if the spot price at the expiration date is \$550?
(b) What would be the payoff to the buyer if the spot price at the expiration date is \$450?

Solution.
(a) Since the spot price is larger than the strike price, the buyer will most likely exercise his option. The payoff in this case is

$$\max\{0, 550 - 500\} = \$50.$$

(b) In this case, the buyer will not exercise his option and therefore the payoff is

$$\max\{0, 450 - 500\} = \$0 \blacksquare$$

Notice that the payoff does not take into consideration the cost of acquiring the position, i.e. the premium which is paid at the time the option is acquired. The payoff of an option is not the money earned (or lost). The **profit** earned by the buyer is given by the formula

$$\text{Buyer's call profit} = \text{Buyer's call payoff} - \text{future value of premium}$$

Example 60.3
Consider a 3-month call option with strike price of \$500 and premium of \$46.90. If the spot price at expiration date is \$550, would you exercise the call option? How much would you make or lose? The risk-free interest rate is 1% over three months.

Solution.
Since spot price is higher than the strike price, you expect to exercise the call option. The future value of the premium is 46.90(1.01) = \$47.37. The buyer's profit is then

$$\max\{0, 550 - 500\} - 47.37 = \$2.63 \blacksquare$$

A payoff diagram and a profit diagram of a long call with strike price K, spot price P_T, and future value of premium P_c is shown in Figure 60.1.

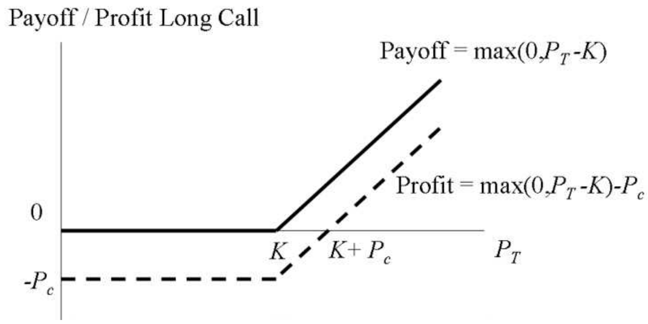


Figure 60.1

Notice that for a long call the maximum loss is the future value of the premium whereas the maximum gain is unlimited.

Example 60.4
Consider a long call option (with strike price K) and a long forward contract (with a forward price K) with underlying asset a stock index. Draw the profit diagram of both on the same window.
(a) If the stock index is rising past K,, which of the two is more profitable?
(b) If the stock index falls considerably, which is more profitable?

Solution.
Figure 60.2 plots the profit for both the long forward and the long call.

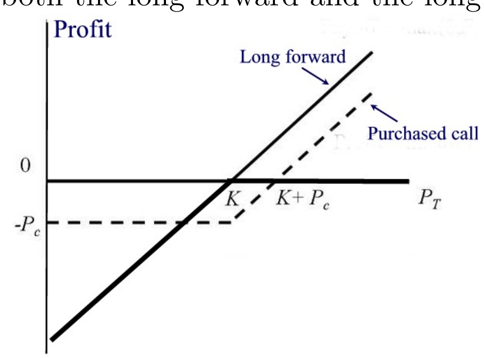


Figure 60.2

(a) The forward contract is more profitable than the call option.
(b) The call option is more profitable (i.e. less loses) because the most you lose is the future

value of your premium. Notice that a call option can be thought as an *insured* position in the index. It protects the buyer against substantial loses. The cost of this insurance is the premium paid ■

Thus far we have considered the payoff and the profit from the buyer's perspective. Next, we consider these issues from the seller's (i.e. the **writer**) perspective.
The writer's payoff and profit are the negative of those of the buyer's and thus they are given by

$$\begin{aligned}\text{Writer's call payoff} &= -\max\{0, \text{spot price at expiration} - \text{strike price}\}\\ &= \min\{0, \text{strike price} - \text{spot price at expiration}\}\end{aligned}$$

and

$$\text{Writer's call profit} = \text{Writer's call payoff} + \text{future value of premium.}$$

Example 60.5
Again, consider a 3-month call option with a strike price of \$500 and a premium \$46.90. Find the writer's payoff and the profit if the spot price at the expiration date is \$550. Assume a risk-free 3-month interest rate of 1%.

Solution.
Since the spot price is higher than the strike price, the writer has to sell the option. His/her payoff is

$$\min\{0, 500 - 550\} = -\$50$$

and the profit is

$$\min\{0, 500 - 550\} + 46.90(1.01) = -\$2.63 \blacksquare$$

The payoff and profit diagrams for a short call with strike price K, future value of premium P_c are shown in Figure 60.3.

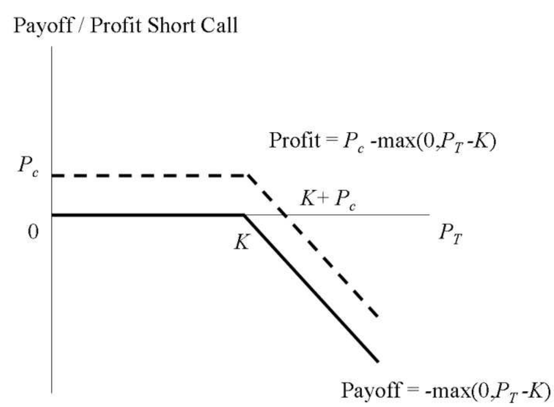


Figure 60.3

Note that for a short call, the maximum gain is the future value of the premium whereas the maximum loss can be unlimited.

Example 60.6
Consider a short call option (with strike price K) and a short forward contract (with a forward price K) with underlying asset a stock index. Draw the profit diagram of both on the same window.
(a) If the stock index is rising, which of the two is more profitable?
(b) If the stock index falls considerably, which is more profitable?

Solution.
Figure 60.4 plots the profit on both the short forward and the short call.

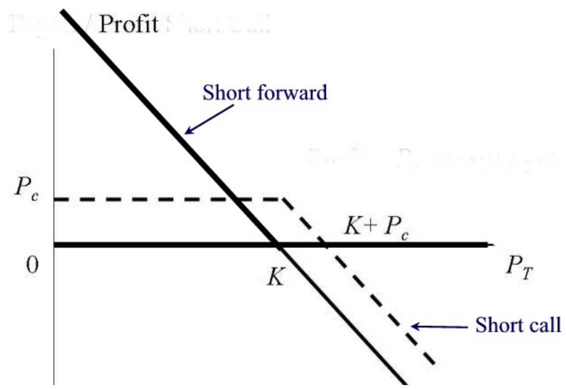


Figure 60.4

(a) Both derivatives loss when stock price rises. However, the loss with the short forward is higher than that with the short call.
(b) The short forward is more profitable than the short call ■

Practice Problems

Problem 60.1
A person who buys an option may do any of the following except
(A) extend it.
(B) exercise it.
(C) sell it.
(D) allow it to expire.

Problem 60.2
Define the following terms associated with options:
(a) Option.
(b) Exercise.
(c) Strike price.
(d) Expiration date.
(e) Call option.

Problem 60.3
What is the difference between European options and American options?

Problem 60.4
An investor buys a call option whose underlying asset is a stock index. The strike price is \$1020 and the expiration date is six months.
(a) If the spot price at the expiration date is \$1100, would the buyer exercise his option? What is his payoff in this case?
(b) What if the spot price is \$900?

Problem 60.5
An investor buys a call option whose underlying asset is a stock index. The strike price is \$1020 and the expiration date is six months.
(a) If the spot price at the expiration date is \$1100, what would the seller's payoff be?
(b) If the spot price at the expiration date is \$900, what would the seller's payoff be?

Problem 60.6
Consider a call option (whose underlying asset is a stock index) with strike price of \$1000 and expiration date in six months. Suppose the risk-free 6-month interest rate is 2% and the call premium is \$93.81
(a) What the future value of the premium?
(b) What is the buyer's profit if the spot price at the expiration date is \$1100?
(c) What if the spot price at the expiration date is \$900?

Problem 60.7
Graphically, the profit diagram of a call option is a vertical shift of the payoff diagram by what quantity?

Problem 60.8
Suppose you buy a 6-month call option with a strike price of \$50. What is the payoff in 6 months for prices \$40, \$45, \$50, \$55, and \$60?

Problem 60.9
A call option on ABC Corp stock currently trades for \$6. The expiration date is December 17,2005. The strike price of the option is \$95.
(a) If this is an American option, on what dates can the option be exercised?
(b) If this is a European option, on what dates can the option be exercised?
(c) Suppose the stock price on Nov 14, 2005 is \$80. Is this option worthless?

Problem 60.10
The strike price of a call option on Sony Corporation common stock is \$40.
(a) What is the payoff at expiration on this call if, on the expiration date, Sony stock sells for \$35?
(b) What is the payoff at expiration of this call if, on the expiration date, Sony stock sells for \$45?
(c) Draw the payoff diagram for this option.

Problem 60.11
You hold a European call option contract (i.e. 100 shares) on Coca-Cola stock. The exercise price of the call is \$50. The option will expire in moments. Assume there are no transactions costs or taxes associated with this contract.
(a) What is your profit on this contract if the stock is selling for \$51?
(b) If Coca-Cola stock is selling for \$49, what will you do?

Problem 60.12
Suppose the stock price is \$40 and the effective annual interest rate is 8%.
(a) Draw the payoff diagrams on the same window for the options below.
(b) Draw the profit diagrams on the same window for the options below.
(i) 35-strike call with a premium of \$9.12.
(ii) 40-strike call with a premium of \$6.22.
(iii) 45-strike call with a premium of \$4.08.

Problem 60.13
What will be the value of adding a long call payoff with a short call payoff?

Problem 60.14
What position is the opposite of a purchased call option?

Problem 60.15
What is the difference between entering into a long forward contract when the forward price is \$55 and taking a long position in a call option with a strike price of \$55?

Problem 60.16
The current spot price of a market index is \$900. An investor buys a 3-month long call option on the index with strike price at \$930. If the spot price at the expiration date is \$920, what is the investor's payoff?

Problem 60.17
You would like to speculate on a rise in the price of a certain stock. The current stock price is \$29, and a 3-month call with a strike price of \$30 costs \$2.90. You have \$5,800 to invest. Identify two alternative investment strategies, one in the stock and the other in an option on the stock. What are the potential gains and losses from each?

Problem 60.18
An American call option allows the holder to:
(A) buy the underlying asset at the strike price on or before the expiration date.
(B) sell the underlying asset at the strike price on or before the expiration date.
(C) buy or sell the underlying asset at the strike price only on the expiration date.
(D) None of the above.

Problem 60.19
Which of the following is TRUE regarding the purchaser of a call option?
(A) The yield on the purchasers portfolio would decrease by purchasing the option
(B) The purchaser would limit the amount of money he could lose if the underlying stock declined
(C) The purchaser would benefit if the underlying stock declined
(D) The purchaser would exercise the option if the stock declined

Problem 60.20
Which of the following statements about the payoff of a call option at expiration is false?
(a) A short position in a call option will result in a loss if the stock price exceeds the exercise price.
(b) The payoff of a long position equals zero or the stock price minus the exercise price, whichever is higher.
(c) The payoff of a long position equals zero or the exercise price minus the stock price, whichever is higher.
(d) A short position in a call option has a zero payoff for all stock prices equal to or less than the exercise price.

Problem 60.21
Suppose that the strike price of a call option is \$80 and future value of premium at expiration is \$7. Complete the following table.

Spot price at expiration	Payoff to buyer	Payoff to seller	Profit/Loss to buyer	Profit/Loss to seller
50				
60				
70				
80				
90				
100				
110				

Problem 60.22
The strike price of an European call option is 88 and the cost (including interest) is \$3.50. For what value of the stock at maturity will the option buyer exercise the option? For what spot prices at maturity the option buyer makes a profit?

Problem 60.23
A European call option on Intel ordinary shares with exercise price \$10.50 per share expires on June 30:
(A) Permits the option writer to sell Intel shares for \$10.50 on June 30.
(B) Requires the option writer to buy Intel shares for \$10.50 any time before or on June 30, at the discretion of the option holder.
(C) Permits the option holder to buy Intel shares for \$10.50 on June 30.
(D) Permits the option holder to sell Intel shares for \$10.50 on June 30.

61 Put Options: Payoff and Profit Diagrams

A call option gives the right to the option holder to buy or walk away. A **put option** gives the right, but not the obligation, to the option holder (the put buyer) to sell the underlying asset at the strike price to the option writer. Note that the buyer of a put option is the seller of the underlying asset. The put seller is obligated to buy the underlying asset if the put is exercised.
In a put option, the put buyer thinks price of an asset will decrease. The buyer pays an upfront premium which he will never get back. The buyer has the right to sell the asset (to the put seller) at strike price. The put seller (or option writer) receives a premium. If buyer exercises the option, the option writer will buy the asset at strike price. If buyer does not exercise the option, the option writer's profit is just the premium.
For a put option, the put buyer has a short position (since he has the right to sell the underlying asset) whereas the put seller has a long position (since he has the obligation to buy the underlying asset from the put holder).

Example 61.1
Consider a put option with underlying asset a stock index and strike price of \$500 in 3 months.
(a) If in three months the stock index is \$550, would the option holder exercise the option?
(b) What if the stock index is \$450?

Solution.
(a) If the stock index is rising the option holder will not exercise the option and he will walk away.
(b) Since the stock index is below \$500, the option holder will sell the stock and earn \$50 ■

Now, since a buyer of a put option will exercise the option only when the spot price is lower than the strike price, the buyer's payoff is then defined by the formula

$$\text{Buyer's put payoff} = \max\{0, \text{strike price} - \text{spot price at expiration}\}.$$

Example 61.2
Consider a put option with a strike price of \$500.
(a) What would be the payoff to the buyer if the spot price at the expiration date is \$550?
(b) What would be the payoff to the buyer if the spot price at the expiration date is \$450?

Solution.
(a) Since the spot price is larger than the strike price, the buyer will not exercise his option. The payoff in this case is

$$\max\{0, 500 - 550\} = \$0.$$

(b) In this case, the buyer will exercise his option and therefore the payoff is

$$\max\{0, 500 - 450\} = \$50 \blacksquare$$

Note that the payoff does not take into consideration the cost of acquiring the position, i.e. the premium which is paid at the time the option is acquired. The **profit** earned by the buyer is found by the formula

$$\text{Buyer's put profit} = \text{Buyer's put payoff} - \text{future value of premium}$$

Example 61.3
Consider a put option with strike price of \$500, spot price at expiration date of \$550 and premium \$41.95. Assume the risk-free interest rate for the three months is 1%. Would you exercise the put option? How much would you make or lose?

Solution.
Since spot price is higher than the strike price, you expect not to exercise the put option. In this case your profit is

$$\max\{0, 500 - 550\} - 41.95(1.01) = -\$42.37.$$

That is you will lose $-\$42.37$ $\blacksquare$

The payoff diagram and the profit diagram of a long put with strike price K and future value of premium P_p are given in Figure 61.1. Note that a call profit increases as the value of the underlying asset increases whereas put profit increases as the value of the underlying asset decreases. Also note that for a purchased put the maximum gain is the strike price minus the future value of the premium whereas the maximum loss is the future value of the premium.

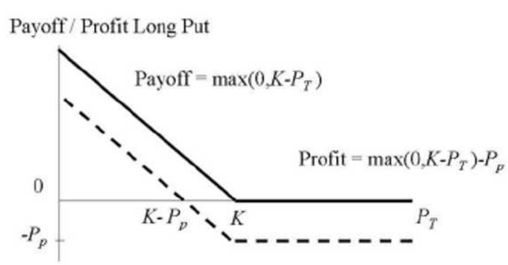


Figure 61.1

Example 61.4
Draw the profit diagram of the put option of the previous example and that of a short forward with delivery price of \$500 and expiration date that of the put option. Which one has higher profit if the index price goes below \$500? What if the index price goes up sufficiently?

Solution.
The profit diagrams for both positions are shown in Figure 61.2. If the index price goes down, the short forward, which has no premium, has a higher profit than the purchased put. If the index goes up sufficiently, the put outperforms the short forward ■

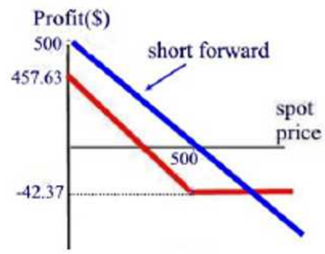


Figure 61.2

A put option is like an insured short forward. When prices go up, the losses from a short forward can be potentially unlimited. With a put the losses are limited.
Up to this point we have considered the payoff and the profit from the buyer's perspective. Next, we consider these issues from the seller's (i.e. the **writer**) perspective.
The writer's payoff and profit are the negative of those of the buyer's and thus they are given by

$$\text{Writer's put payoff} = \min\{0, \text{spot price at expiration} - \text{strike price}\}$$

and

$$\text{Writer's put profit} = \text{Writer's put payoff} + \text{future value of premium}.$$

Example 61.5
Again, consider a put option with a strike price of \$500 and a premium valued at expiration date for \$41.95. Find the writer's payoff and the profit if the spot price at the expiration date is \$550.

Solution.
Since the spot price is higher than the strike price, the writer does not have to buy the underlying asset. His/her payoff is

$$\min\{0, 550 - 500\} = \$0$$

and the profit is

$$\min\{0, 550 - 500\} + 41.95(1.01) = \$42.37 \blacksquare$$

The payoff and profit diagrams of a short put are shown in Figure 61.3.

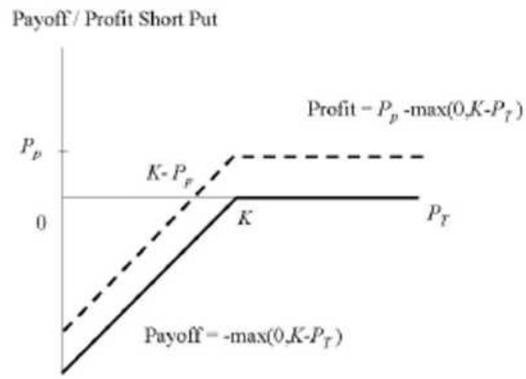


Figure 61.3

Note that for a written put the maximum gain is the future value of the premium whereas the maximum loss is the future value minus the strike price.

Options as Insurance Policies

Options can be used very much like insurance policies against moving prices and a way to protect physical assets already owned. For example, a call option can provide insurance against the rise in the price of something we plan to buy in the future. The long position is buying the insurance and the short position is selling the insurance. A put option can provide insurance against the fall in the price of an underlying asset. The long position is selling the insurance and the short position is buying the insurance.

A homeowner insurance policy is an example of a put option. For example, suppose your house is valued at \$200,000. You buy a homeowner insurance policy for the premium of \$15,000 that expires in one year. The policy requires a deductible for \$25,000. This means that if the damage of the house is less than \$25,000 you are fully responsible for the cost. If the damage is greater than \$25,000 then the insurance company will cover the portion above the \$25,000. The insurance policy acts as a put option. The premium \$15,000 is like the premium of a put, the expiration date is one year, and the striking price of the put is \$175,000.

The Moneyness of an Option

In finance, "moneyness" is a measure of the degree to which a derivative is likely to have positive monetary value at its expiration.

From the buyer perspective and neglecting the premium for buying an option, an option is said to be **in-the-money** if the buyer profits when the option is exercised immediately. For example, a call option with spot price larger than the strike and a put option with strike price larger than a

spot price are both in-the-money.
On the other hand, an option is said to be **out-of-the-money** option if the buyer loses when the option is exercised immediately. A call option with spot price less than the strike price and a put option with strike price less than the spot price are both out-of-the money.
An option is said to be **at-the-money option** if the buyer does not lose or profit when the option is exercised immediately. This occurs when the spot price is approximately equal to the strike price. Figure 61.4 illustrates these concepts.

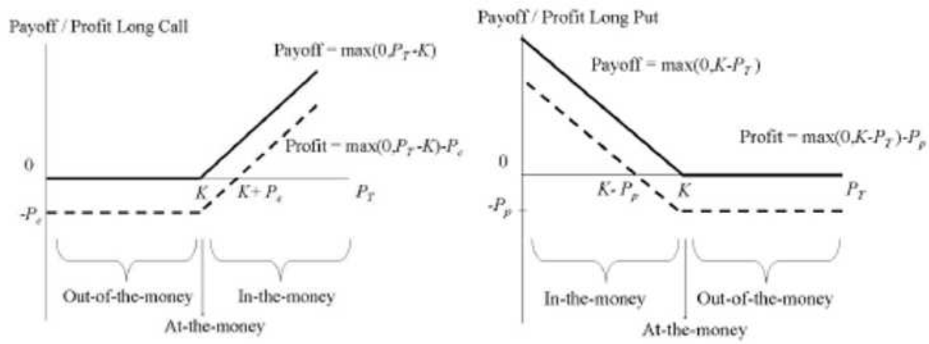


Figure 61.4

Example 61.6
If the underlying stock price is $25, indicate whether each of the options below is in the money, at the money, or out of the money.

Strike	Call	Put
$20		
$25		
$30		

Solution.

Strike	Call	Put
$20	In the money	out of the money
$25	At the money	At the money
$30	Out of the money	In the money

Practice Problems

Problem 61.1
Suppose you have 5,000 shares worth \$25 each. How can put options be used to provide you with insurance against a decline in the value of your holdings over the next four months?

Problem 61.2
Consider a put option whose underlying asset is a stock index with 6 months to expiration and a strike price of \$1000.
(a) What is the buyer's payoff if the index price is \$1100 in 6 months?
(b) What is the buyer's payoff if the index price is \$900 in 6 months?

Problem 61.3
Consider a put option whose underlying asset is a stock index with 6 months to expiration and a strike price of \$1000. Suppose the risk-free interest rate for the six months is 2% and that the option's premium is \$74.20.
(a) Find the future premium value in six months.
(b) What is the buyer's profit is the index spot price is \$1100?
(c) What is the buyer's profit is the index spot price is \$900?

Problem 61.4
A put is trading on Sony Corporation stock. It has a strike price of \$40.
(a) What is the payoff at expiration of this put if, on the expiration date, Sony stock sells for \$45?
(b) What is the payoff at expiration of this call if, on the expiration date, Sony stock sells for \$25?
(c) Draw the payoff diagram for this option.

Problem 61.5
A put option in which the stock price is \$60 and the exercise price is \$65 is said to be
(a) in-the-money
(b) out-of-the-money
(c) at-the-money
(d) none of the above

Problem 61.6
What position is the opposite of a purchased put option?

Problem 61.7
Suppose that you own 5000 shares worth \$30 each.How can put options be used to provide you with insurance against a decline in the value of your holding over the next four months?

Problem 61.8
A Swiss company knows that it is due to receive \$1,400,000 in (exactly) one year. What type of option contract is appropriate for hedging?

Problem 61.9
A trader buys a European call option and sells a European put option. The options have the same underlying asset, strike price, and maturity date. Describe the trader's position.

Problem 61.10
Explain carefully the difference between selling a call option and buying a put option. Assume European options.

Problem 61.11
Suppose that you write a put contract with a strike price of \$40 and an expiration date in 3 months. The current stock price is \$41 and the contract is on 100 shares. What have you committed yourself to? How much could you gain or lose?

Problem 61.12
Describe the moneyness of the following options, i.e. as in-the-money, out-of-the-money or at-the-money:
(a) A call or a put option with spot price of \$100 at expiration and strike price of \$100.
(b) A call option with a spot price of \$100 at expiration and strike price of \$80.
(c) A put option with a spot price of \$100 at expiration and strike price of \$80.
(d) A call option with a spot price of \$100 at expiration and strike price of \$120.
(e) A put option with a spot price of \$100 at expiration and strike price of \$120.

Problem 61.13
Suppose the stock price is \$40 and the effective annual interest rate is 8%.
(a) Draw the payoff diagrams on the same window for the options below.
(b) Draw the profit diagrams on the same window for the options below.
(i) 35-strike put with a premium of \$1.53.
(ii) 40-strike put with a premium of \$3.26.
(iii) 45-strike put with a premium of \$5.75.

Problem 61.14
You are a speculator and you think stock prices will increase. Should you buy a call or a put option ?

Problem 61.15
A call option and a put option of a stock index have an exercise price of \$60. Complete the following table of payoffs.

stock price	Buy call	Write call	Buy put	Write put
\$30				
\$40				
\$70				
\$900				

Problem 61.16
A trader writes a December put option with a strike price of \$30. The premium of the option is \$4. Under what circumstances does the trader make a gain? Ignore the time value of money.

Problem 61.17
Complete the following table.

Derivative Position	Maximum Loss	Minimum Gain	Position with Respect to Underlying Asset	Strategy
Long Forward				
Short Forward				
Long Call				
Short Call				
Long Put				
Short Put				

Problem 61.18
The payoff of an out-of-the-money long put is equal to
A. the stock price minus the exercise price.
B. the put premium.
C. zero.
D. the exercise price minus the stock price.
E. none of the above.

Problem 61.19
An option whose striking price is above the stock price is
A. out-of-the-money.
B. in-the-money.
C. at-the-money.
D. cannot be determined.

Problem 61.20
Options are not normally
A. exercised early.
B. exercised after expiration.
C. exercised when out-of-the-money.
D. All of the above

Problem 61.21
If you write a put option,
A. the maximum profit is unlimited.
B. the maximum loss is unlimited.
C. the maximum gain equals the premium.
D. the maximum gain equals the stock price minus the striking price.

Problem 61.22
If someone writes a put, they usually want the market to
A. go up.
B. go down.
C. stay unchanged.
D. fluctuate.

Problem 61.23
An American put option on Intel ordinary shares with exercise price $10.50 per share expires on June 30:
(A) Permits the option holder to sell Intel shares for $10.50 any time before or on June 30.
(B) Permits the option writer to sell Intel shares for $10.50 on June 30.
(C) Requires the option writer to sell Intel shares for $10.50 on June 30, at the discretion of the option holder.
(D) Requires the option holder to buy Intel shares for $10.50 any time before or on June 30, at the discretion of the option writer.

Problem 61.24
A stock is trading at a price of $20 at expiration date. Which of the following statement about options would be correct?

(i) A put option with a strike of $17.50 would be in-the-money.
(ii) A put option with a strike of $20 would be at-the-money.
(iii) A put option with a strike price of $25 would be in-the-money.

(iv) A call option with a strike price of \$20 would be in-the-money.
(v) A call option with a strike price of \$17.50 would be out-of-the-money.

(A) i, iv and v only.
(B) i, ii, iv and v only.
(C) ii, iv and v only.
(D) ii and iii only.

62 Stock Options

Options can have various underlying securities, namely:
• **Index Options:** An index option contract is where the underlying security is not an individual share but an index (i.e. a collection of stocks) such as the S& P 500 index or the Nasdaq Compsoite index. Thus the buyer of a call option has a right to buy the index at a predetermined price on or before a future date. All index option contracts are cash settled.
• **Futures Options:** An option with underlying asset a futures contract.
• **Foreign currency options:** An option which gives the owner the right to buy or sell the indicated amount of foreign currency at a specified price before or on a specific date.
• **Stock options.** In this section we consider stock options and the main factors that an investor should be aware off when trading with stock options.
A **stock option** is the right to sell or buy a specified number of shares of a stock at a specific price (the strike price) and time (the expiration date.) Stock options are the most popular long-term incentive compensation approach used in U.S. companies.
A **stock option contract** is a contract representing 100 shares in the underlying stock. For example, if you want to own 1000 shares of Microsoft then you either can buy the 1000 shares at the stock exchange or you buy 10 Microsoft option contracts at the options exchange.
A stock option's expiration date can be up to nine months from the date the option is first listed for trading. Longer-term option contracts, called LEAPS (Long-Term Equity AnticiPation Securities), are also available on many stocks, and these can have expiration dates up to three years from the listing date. In practice, stock options expire on the third Saturday of the expiration month but the last day of trading is the third Friday of the expiration month.
Buying and selling stock options are not made directly between the buyer (holder) and the seller (writer) of the stock option but rather through a brokerage firm. In reality, an organization called the OCC or Options Clearing Corporation steps in between the two sides. The OCC buys from the seller and sells to the buyer. This makes the OCC neutral, and it allows both the buyer and the seller to trade out of a position without involving the other party.

Example 62.1
The following is listed in an edition of the Wall Street Journal: IBM Oct 90 Call at \$2.00. Suppose you buy one contract.
(a) What type of option is considered in the listing?
(b) What is the strike price of a share of the stock?
(c) When does the option expires?
(d) If you own the option and you want to sell, how much will you receive?
(e) If you are buying the option, how much will it cost you?
(f) Suppose you hold the option and you decide to exercise, how much will you pay?

Solution.
(a) This is a call stock option.
(b) The strike price is \$90 per share.
(c) It expires on the Saturday following the third Friday of October in the year it was purchased.
(d) You will receive $\$2 \times 100 = \$200-$ commission.
(e) The cost of buying the option is $\$2 \times 100+$ brokerage commission.
(f) You have to pay $\$90 \times 100+$ brokerage commission ■

In this section we consider the basic issues that an investor must be aware of when considering buying stock options, namely, dividends, exercise, margins, and taxes.

The Dividend Effect
Dividends that might be paid during the life of a stock option can affect the stock price. We may regard dividends as a cash return to the investors. The company has the choice to either paying dividends to the shareholders or reinvesting that money in the business. The reinvestment of that cash could create more profit for the business and therefore leading to an increase in stock price. On the other hand, paying dividends effectively reduce the stock price by the amount of the dividend payment. Such a drop has an adverse effect on the price of a call option and a beneficial effect on the price of a put option. Since the payoff of a call option is the maximum of zero or the stock price minus the exercise price, a drop in the stock price will lower the payoff of a call option at expiration date. On the contrary, since the payoff of a put option is the maximum of zero or the exercise price minus the stock price, lowering the stock price will increase the return on a put option.

Example 62.2
Consider a call option and a put option with the same expiration date and the same strike price of \$100.
(a) Assume no dividends occurred during the life of the options, find the payoff of both options if the stock price is \$102 at expiration.
(b) Suppose that a dividend of \$3 was paid just before expiration. What will be the stock price at expiration?
(c) Find the payoff of both options after the dividend was paid.

Solution.
(a) The payoff of the call option is

$$\max\{0, 102 - 100\} = \$2$$

while that of the put option is

$$\max\{0, 100 - 102\} = \$0.$$

(b) The stock price will decline by \$3 so that the stock price at expiration is \$99.
(c) The call option's payoff is

$$\max\{0, 99 - 100\} = \$0$$

and the put's option payoff is

$$\max\{0, 100 - 99\} = \$1 \blacksquare$$

The Exercising Effect
After buying an option, an investor must provide exercise instructions to the broker's deadline. Failing to do so, the option will expire worthless. However, this is not required for cash-settled options where the option is automatically exercised at the expiration date.
Exercising an option usually results in a commission paid to the broker. Exercising a put option results in an additional commission for selling the underlying shares. Therefore, if a holder of a stock option does not wish to own the shares of the stock he is better of selling the option instead of exercising it. When an option is exercised the writer of the option is obligated to sell the underlying assets (in the case of a call option) or buy the underlying assets (in the case of a put option). He is said to have been **assigned**. Assignment results in a commision to be paid by the option's writer.
Dividends are one factor that can affect the exercise decision. Since American stock options can be exercised at anytime before the expiration date, dividends make early exercise more likely for American calls and less likely for American puts.

Example 62.3
An investor buys a European put on a share for \$3. The strike price is \$40.
(a) Under what circumstances will the option be exercised?
(b) Under what circumstances does the investor make a profit?
(c) Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

Solution.
(a) The option will be exercised if the stock price is less than \$40 on the expiration date.
(b) The investor makes a profit if the stock price on the expiration date is less than \$37, because the gain from exercising the option is greater than \$3. Taking into account the initial cost of the option (\$3), the profit will be positive.
(c) The graph of the investor's profit as a function of the stock price at the maturity of the option is shown in Figure 62.1.

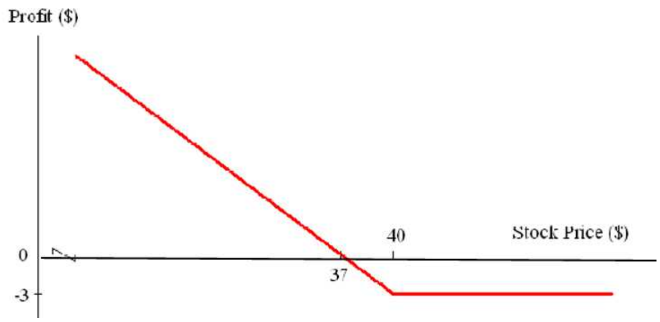


Figure 62.1

Margins for Written Options
By a **margin** we mean a collateral deposited into a margin account to insure against the possibility of default. The writer of an option is in a short position. Thus, an investor is required to post a collateral with his brokerage firm to insure against the possibility of default (i.e. not being able to fulfill his obligations toward the option buyer).
Writers of options should determine the applicable margin requirements from their brokerage firms and be sure that they are able to meet those requirements in case the market turns against them.

Example 62.4
Explain why brokers require margins when the clients write options but not when they buy options.

Solution.
When an investor buys an option, he must pay it upfront. There are no future liabilities and, therefore, no need for a margin account. When an investor writes an option, there are potential future liabilities. To protect against the risk of a default, margins are required ■

Tax Considerations when Trading Stock Options
Tax rules for derivatives are very complicated and change very frequently. Briefly put, under current IRS rules, both stock and stock options are capital assets, and any net gain (i.e. gross gain − costs) on their sale is taxable. If the stock or option was held for less than a year, then **ordinary income tax rates** apply to the gain. But if the stock or option has been held for more than a year, the holder gets a tax break on the sale, because the gain is taxed at the **long-term capital gains rate**. Since all stock options (except LEAPS options) have a duration of 9 months or less you cannot hold them for a year or more, and so tax on option gains will usually be at the short-term (ordinary income tax) rates. All short-term gains and losses are combined on the investor tax return.
One category of tax rules that can apply when trading with stocks is the one that governs constructive sales. If you own a stock and you have entered into a transaction such as a short sale or a

forward contract hoping to reduce your risk from a fluctuating market, the IRS consider that you have made a sale referred to as a **constructive sale** and as a result your gain or loss are subject to taxes. We illustrate this concept in the following example.

Example 62.5
Suppose you own 100 shares of ABC stocks that you bought for $20 per share. So, you have a $2,000 basis in your stock. Currently, ABC stock has reached a peak of $50 a share. So you decide to make a short sale. You borrow 100 shares from a broker and sell them for $50 per share, getting a total of $5,000. Now, you have to give 100 shares back to your broker to close the transaction. So, if the stock price goes up between the time that you sold the borrowed shares and the time that you must give the broker back 100 shares, you just give the broker the 100 shares that you originally own and you are done with it. You have made $3000. If, instead, the stock price goes down in between the time you sold the borrowed shares and the day you must give the broker back his 100 shares you would not give him your original 100 shares, but instead go out to the open market and buy 100 shares and give those new ones to the broker. In this scenario, then, you would make a profit and he would still have your shares to hold onto for long-term. In both scenarios, your gain is taxable.

Practice Problems

Problem 62.1
Assuming you own XYZ company shares trading at \$40 right now. You bought a put stock options contract that allows you to sell your XYZ shares at \$40 anytime before it expires in 2 months. 1 month later, XYZ company shares are trading at \$30. If you decide to exercise the option, at what price will you be selling each share?

Problem 62.2
Explain the meaning of "XYZ April 25 Call."

Problem 62.3
Which is more expensive a paying dividend call option or a paying dividend put option?

Problem 62.4
An investor buys a European call on a share for \$4. The strike price is \$50.
(a) Under what circumstances will the option be exercised?
(b) Under what circumstances does the investor make a profit?
(c) Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

Problem 62.5
A stock price is S just before a dividend D is paid. What is the price immediately after the payment? (Assume that income tax on dividend income is zero.)

Problem 62.6
When would be the best time to exercise an American call on a dividend-paying stock?

Problem 62.7
A company has 400,000 shares and the stock price is currently \$40 per share. If the company issues 25% stock dividend (i.e. for every 100 shares of stock owned, 25% stock dividend will yield 25 extra shares), what would you expect the stock price to be after the dividend is paid?

Problem 62.8
In January, 1995, you bought 100 shares of XYZ stock at \$5 a share. One day in March, 1995, when the stock was worth \$10 a share, you borrowed 100 shares from your broker, sold those, and pocketed the \$1000. You had, so far, made \$500. But eventually you would have to give the stocks back to the broker.
(a) Early in 1996, the stock went up to \$15 a share. What would you do to give the broker back his shares?
(b) Are you supposed to report the gain to the IRS?

Problem 62.9
A LEAP is a
A. commodity option.
B. gold or silver option.
C. long-term option.
D. option with a high striking price.

Problem 62.10
The guarantor of option trades is the
A. SEC
B. OCC
C. CFTC
D. FDIC

Problem 62.11
LEAPS are issued with durations of all of the following except
A. 1 year
B. 2 years
C. 3 years
D. 10 years

Problem 62.12
When is the last trading day for an option?
(A) The last business day of the expiry month.
(B) The last Friday of the expiry month.
(C) The first business day of the expiry month.
(D) The third Friday of the expiry month.

63 Options Strategies: Floors and Caps

In this section we discuss the strategy of using options to insure assets we own (or purchase) or assets we short sale.

Insuring Owned Assets with a Purchased Put: Floors
An investor who owns an asset (i.e. being long an asset) and wants to be protected from the fall of the asset's value can insure his asset by buying a put option with a desired strike price. This combination of owning an asset and owning a put option on that asset is called a **floor**. The put option guarantees a minimum sale price of the asset equals the strike price of the put.
To examine the performance of this strategy, we look at the combined payoff and profit of the asset position and the put as illustrated in the example below.

Example 63.1
Suppose you buy an index valued at \$500. To insure your index from value decline you buy a 500-strike 3-month put with risk-free 3-month rate of 1% and a premium of \$41.95.
(a) Complete the table below where the values are assumed at the expiration date.

Index Payoff	Put Payoff	Combined Payoff	Combined Profit
400			
450			
500			
550			
600			
650			
700			

(b) Graph the payoff and profit diagrams for the combined position.
(c) Compare the payoff diagram of the combined position with the payoff of a long 500-strike 3-month call with risk-free 3-month interest of 1% and a premium of \$46.90.
(d) Compare the profit diagram of the combined position with the profit diagram of a long 500-strike 3-month call with risk-free 3-month rate of 1% and a premium of \$46.90.
(e) Compare the payoff and profit diagrams of the combined position with the payoff and profit diagrams of the combined position of a 3-month 500-strike call (with 3-month risk-free interest of 1% and premium 0f \$46.90) and a zero-coupon bond that pays \$500 in three months (with present cost of $500(1.01)^{-1} = \$495.05$).

Solution.
Note that the profit of the combined position is the profit from the long index plus the profit from the long put. That is, the combined profit is

$$\text{Index payoff} - 500 \times 1.01 + \text{Put payoff} - 41.95 \times 1.01 = \text{Combined payoff} - \$547.37.$$

(a) Recall that the (long) index payoff is the spot price of the index and the (long) put payoff is equal to $\max\{0, \text{strike price} - \text{spot price}\}$.

Index Payoff	Put Payoff	Combined Payoff	Combined Profit
400	100	500	-47.37
450	50	500	-47.37
500	0	500	-47.37
550	0	550	2.63
600	0	600	52.63
650	0	650	102.63
700	0	700	152.63

(b) Figure 63.1 shows the payoff and the profit diagrams of the combined position.

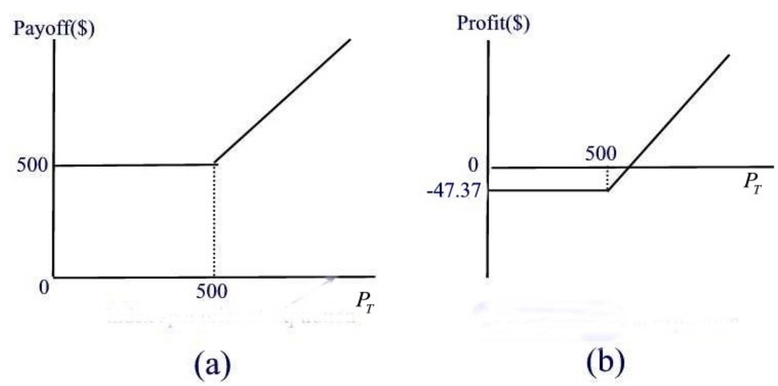


Figure 63.1

Notice that the level of the floor is $-\$47.37$, which is the lowest possible profit (a loss in this case).
(c) We notice from Figure 63.1(a) that the payoff diagram of the combined position is a vertical shift of the long call (compare with Figure 60.1).
(d) The profit diagram in Figure 63.1(b) is identical to the profit diagram of a 3-month 500-strike call with 3-month risk-free interest of 1% and premium of \$46.90 (See Figure 60.1). It follows that the cash flows in purchasing a call are different from the cash flows of buying an asset and insuring it with a put, but the profit for the two positions is the same.
(e) Recall that a long call payoff is equal to $\max\{0, \text{spot price} - \text{strike price}\}$. Also, the combined profit is

$$\text{Call payoff} - 46.90 \times 1.01 + \text{Bond payoff} - 495.05 \times 1.01 = \text{Combined payoff} - \$547.37$$

We have the following table for the combined position of the 3-month 500-strike call with a zero-coupon bond that pays \$500 in three months.

spot Price	Call Payoff	Bond's Payoff	Combined Payoff	Combined Profit
400	0	500	500	−47.37
450	0	500	500	−47.37
500	0	500	500	−47.37
550	50	500	550	2.63
600	100	500	600	52.63
650	150	500	650	102.63
700	200	500	700	152.63

Thus, the combined position of index plus put is equivalent to the combined position of a zero-coupon bond plus call ■

Insuring a Short Sale: Caps

When you short an asset, you borrow the asset and sell, hoping to replace them at a lower price and profit from the decline. Thus, a short seller will experience loss if the price rises. He can insure his position by purchasing a call option to protect against a higher price of repurchasing the asset. This combination of short sale and call option purchase is called a **cap**.

To examine the performance of this strategy, we look at the combined payoff and profit of the asset position and the call as illustrated in the example below.

Example 63.2

Suppose you short an index valued at \$500. To insure your index from value increase you buy a 500-strike 3-month call with risk-free 3-month rate of 1% and a premium of \$46.90.

(a) Complete the table below where the values are assumed at the expiration date.

Short Index Payoff	Call Payoff	Payoff	Accumulated Cost	Profit
−400				
−450				
−500				
−550				
−600				
−650				
−700				

(b) Graph the payoff and profit diagrams for the combined position.

(c) Compare the payoff diagram of the combined position with the payoff diagram of a purchased 500-strike put option with expiration date of three months, risk-free 3-month interest of 1% and

premium \$41.95.
(d) Compare the profit diagram of the combined position to the profit diagram of a 500-strike put with expiration date of three months, risk-free 3-month interest of 1% and premium of \$41.95.
(e) Compare the payoff and profit diagram of the combined position with the payoff and profit diagrams of a purchased 500-strike put option with expiration date of three months, risk-free 3-month interest of 1% and premium \$41.95 coupled with borrowing $500(1.01)^{-1} = \$495.05$.

Solution.
(a) Recall that the short index payoff is the negative of the long index position. Also, recall that (long) call payoff is $\max\{0, \text{spot price} - \text{strike price}\}$. The profit of the combined position is the profit from the short index plus the profit from the long call. That is, the combined profit is

$$-\text{ long index payoff} + 500 \times 1.01 + \text{long call payoff} - 46.90 \times 1.01 = \text{Combined payoff} + \$457.63$$

Short Index Payoff	Call Payoff	Combined Payoff	Combined Profit
−400	0	−400	57.63
−450	0	−450	7.63
−500	0	−500	−42.37
−550	50	−500	−42.37
−600	100	−500	−42.37
−650	150	−500	−42.37
−700	200	−500	−42.37

(b) Figure 63.2 shows the payoff and the profit diagrams of the combined position.

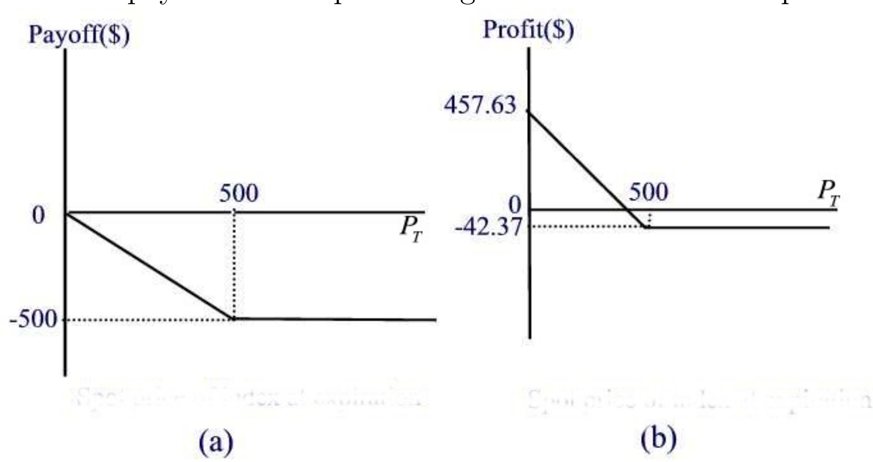


Figure 63.2

(c) By comparing Figure 63.2 with Figure 61.1, we note that the payoff diagram of the combined position generates a position that looks like a long put.
(d) The profit diagram in Figure 63.2(b) is identical to the profit diagram of a 3-month 500-strike put with 3-month risk-free interest of 1% and premium of \$41.95 (See Figure 61.2).
(e) Recall that a long put payoff is equal to $\max\{0, \text{strike price} - \text{spot price}\}$ and the loan payoff is $-\$500$ so that the combined payoff is $\max\{0, \text{strike price} - \text{spot price}\} - 500$. Since borrowing has no effect on the profit, the combined profit is just

$$\text{Put payoff } -41.95 \times 1.01 = \text{Put payoff } -\$42.37$$

We have the following table for the combined position of the 3-month 500-strike put with borrowing \$495.05.

Spot Price	Put Payoff	Combined Payoff	Combined Profit
400	100	−400	57.63
450	50	−450	7.63
500	0	−500	−42.37
550	0	−500	−42.37
600	0	−500	−42.37
650	0	−500	−42.37
700	0	−500	−42.37

Thus, the combined position of short sale plus call is equivalent to the combined position of a purchased put coupled with borrowing ■

Practice Problems

Problem 63.1
Suppose a speculator is short shares of stock. Which of the following would be the best hedge against the short position?
A. buying a put
B. writing a put
C. buying a call
D. writing a call

Problem 63.2
Buying a long position in the underlying asset and a long put resembles
(A) Buying a long position in the underlying asset and a long call
(B) Buying a long call
(C) Buying a short call
(D) Buying a long position in the underlying asset and short call.

Problem 63.3
Shorting an index and a long call resembles
(A) Buying a long position in the underlying asset and a long call
(B) Buying a long call
(C) Buying a long put
(D) Shorting the index and buying and long put.

Problem 63.4
Consider the following combined position:
• buy an index for $500
• buy a 500-strike put with expiration date in 3 months, 3-month risk free rate of 1% and premium of $41.95
• borrow $495.05 with 3-month interest rate of 1%.
Graph the payoff diagram and profit diagram of this position.

Problem 63.5
Suppose you buy an index valued at $1000. To insure your index from value decline you buy a 1000-strike 6-month put with risk-free 6-month rate of 2% and a premium of $74.20.
(a) Construct the payoff diagram and the profit diagram of this combined position.
(b) Verify that you obtain the same payoff diagram and profit diagram by investing $980.39 in zero coupon bond and buying a 1000-strike 6-month call with risk-free 6-month rate of 2% and a premium of $93.81.

Problem 63.6
Suppose you short an index valued for \$1000. To insure your index from value increase you buy a 1000-strike 6-month call with risk-free 6-month rate of 2% and a premium of \$93.81.
(a) Graph the payoff and profit diagrams for the combined position.
(b) Verify that you obtain the same payoff diagram and profit diagram by purchasing a 1000-strike put option with expiration date of six months, risk-free 6-month interest of 2% and premium \$74.20 coupled with borrowing \$980.39.

64 Covered Writings: Covered Calls and Covered Puts

Writing an option backed or covered by the underlying asset (such as owning the asset in the case of a call or shorting the asset in the case of a put) is referred to as **covered writing** or **option overwriting**. The most common motivation for covered writing is to generate additional income by means of premium. In contrast to covered writing, when the writer of an option has no position in the underlying asset we refer to this as **naked writing**. In this section we discuss two types of covered writings: Covered calls and covered puts. Note that covered writing resembles selling insurance.

Writing a covered call
A **covered call** is a call option which is sold by an investor who owns the underlying assets. An investor's risk is limited when selling a covered call since the investor already owns the underlying asset to cover the option if the covered call is exercised. By selling a covered call an investor is attempting to capitalize on a neutral or declining price in the underlying stock. When a covered call expires without being exercised (as would be the case in a declining or neutral market), the investor keeps the premium generated by selling the covered call. The opposite of a covered call is a naked call, where a call is written without owned assets to cover the call if it is exercised.

Example 64.1
Suppose you buy an index valued for \$500 and sell a 500-strike 3-month call with risk-free 3-month rate of 1% and a premium of \$46.90.
(a) Construct the payoff diagram and the profit diagram of this covered call.
(b) Verify that you obtain the same profit diagram by selling a 500-strike 3-month put with risk-free 3-month rate of 1% and a premium of \$41.95.

Solution.
(a) Recall that the long index payoff is the same as the spot index at expiration. Also, recall that the written call payoff is $-\max\{0, \text{spot price} - \text{strike price}\}$. The profit of the combined position is the profit from the long index plus the profit from the written call. That is, the combined profit is

$$\text{Long index payoff } -500 \times 1.01 - \max\{0, \text{spot price} - \text{strike price}\} + 46.90 \times 1.01 = \text{Combined payoff - \$457.63}$$

Long Index Payoff	Written Call Payoff	Combined Payoff	Combined Profit
400	0	400	-57.63
450	0	450	-7.63
500	0	500	42.37
550	-50	500	42.37
600	-100	500	42.37
650	-150	500	42.37
700	-200	500	42.37

Figure 64.1 shows the payoff and the profit diagrams of the combined position.

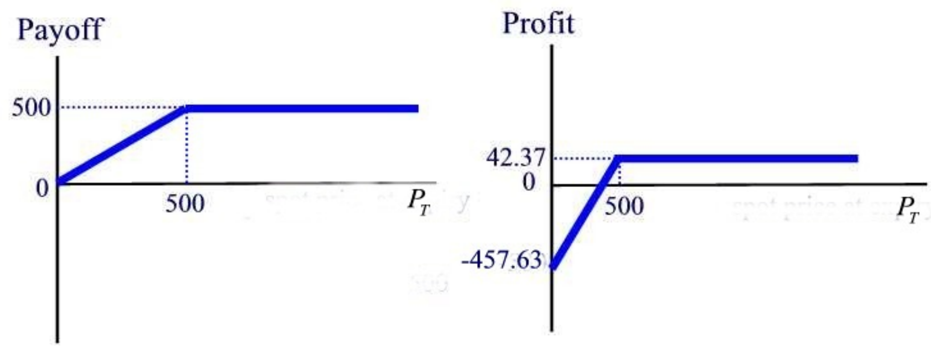


Figure 64.1

(b) Recall that the profit of a written put is $-\max\{0, \text{strike price} - \text{spot price}\} + 41.95 \times 1.01$. The profit diagram is given in Figure 64.2.

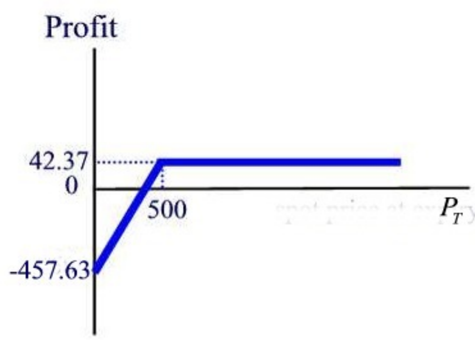


Figure 64.2

Thus selling a covered call has the same profit as selling the put in which we don't own the underlying asset ■.

Writing a covered put

A **covered put** is a put option which is sold by an investor and which is covered (backed) by a short sale of the underlying assets. A covered put may also be covered by deposited cash or cash equivalent equal to the exercise price of the covered put. Since a covered put is covered (or backed) in advance by assets or cash, a covered put represents a known and limited risk should the holder of the put choose to exercise the option of the covered put. The opposite of a covered put would be an uncovered or naked put.

Example 64.2

An investor shorts an index for \$500 and sell a 500-strike put with expiration date in 3 months,

3-month risk free rate of 1% and premium of \$41.95
(a) Graph the payoff diagram and profit diagram of this covered put.
(b) Verify that the profit diagram coincides with the profit diagram of a 500-strike written call with expiration date in 3 months, 3-month risk free rate of 1% and premium of \$46.90.

Solution.
(a) The combined payoff is

$$-\text{ Long index payoff} - \max\{0, \text{strike price} - \text{spot price}\}$$

The combined profit is

$$-\text{ Long index payoff} + 500 \times 1.01 - \max\{0, 500 - \text{spot price}\} + 41.95 \times 1.01$$

or

$$-\text{ Long index payoff} - \max\{0, 500 - \text{spot price}\} + \$547.37$$

The following table summarizes the combined payoff and the combined profit.

Short Index Payoff	Written Put Payoff	Combined Payoff	Combined Profit
−400	−100	−500	47.37
−450	−50	−500	47.37
−500	0	−500	47.37
−550	0	−550	−2.63
−600	0	−600	−52.63
−650	0	−650	−102.63
700	0	−700	−152.63

The payoff diagram and the profit diagram are shown in Figure 64.3.

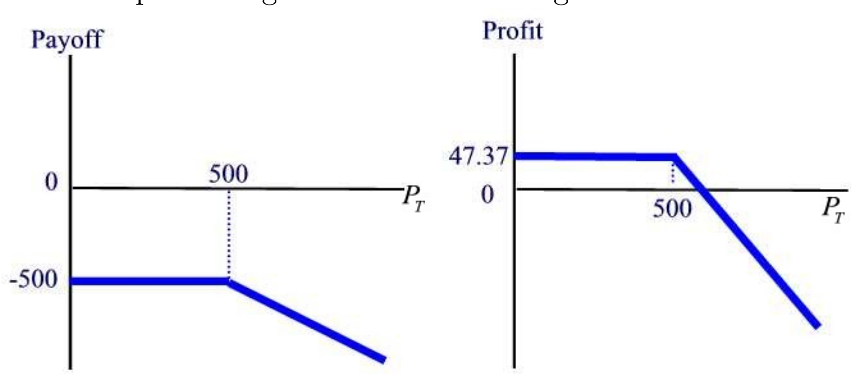


Figure 64.3

(b) Recall that the profit of a written call is given by $-\max\{0, \text{spot price at expiration} - \text{strike price}\} + 46.90(1.01) = -\max\{0, \text{spot price at expiration} - \text{strike price}\} + 47.37$. The profit diagram coincides with the one in Figure 64.3 ■

Example 64.3
An investor sells short 100 shares of ABC stock at \$9.25 a share. He sells one put contract (100 shares) with a striking price of \$10 and a premium of \$1.50 a share.
(a) What is the investor's profit if the spot price is below \$10 at expiration?
(b) The investor's will be making profit as long as the spot price is less than what value at expiration?

Solution.
(a) If the spot price at expiration is below \$10 the put holder will exercise the option. In this case, the investor will buy the shares for \$10 a share and give it back to the lender. The investor's profit is then $9.25 \times 100 + 100 \times 1.50 - 10 \times 100 = \$75$.
(b) The investor will be making profit as long as the spot price at expiration is less than \$10.75 ■

Practice Problems

Problem 64.1
If someone writes a call while owning the underlying asset, the call is
(A) covered
(B) long
(C) naked
(D) cash-secured

Problem 64.2
A covered call position is
(A) the simultaneous purchase of the call and the underlying asset.
(B) the purchase of a share of stock with a simultaneous sale of a put on that stock.
(C) the short sale of a share of stock with a simultaneous sale of a call on that stock.
(D) the purchase of a share of stock with a simultaneous sale of a call on that stock.
(E) the simultaneous purchase of a call and sale of a put on the same stock.

Problem 64.3
A covered put position is
(A) the simultaneous purchase of the put and the underlying asset.
(B) the short sale of a share of stock with a simultaneous sale of a put on that stock.
(C) the purchase of a share of stock with a simultaneous sale of a call on that stock.
(D) the simultaneous purchase of a call and sale of a put on the same stock.

Problem 64.4
If a person writes a covered call with a striking price of \$45 and receives \$3 in premium, exercise will occur if the stock price is above ___ on expiration day.
(A) \$42
(B) \$45
(C) \$48
(D) \$50

Problem 64.5
If a stock is purchased at \$50 and a \$55 call is written for a premium of \$2, the maximum possible gain per share is (neglecting interest)
(A) \$2
(B) \$5
(C) \$7
(D) \$10

Problem 64.6
If someone writes a covered call, they usually want the market to
(A) go up.
(B) go down or stay unchanged.
(C) fluctuate.

Problem 64.7
The most common motivation for option overwriting is
(A) risk management.
(B) tax reduction.
(C) leverage.
(D) income generation.

Problem 64.8
An investor buys 500 shares of ABC stock at \$17 a share. He sells 5 call contracts (100 shares each) with a striking price of \$17.50 and premium of \$2 a share.
(a) What is the investor's initial investment?
(b) What will be the investor's profit at expiration if the spot price of the stock is \$17? Ignore interest.
(c) What will be the investor's profit at expiration if the spot price of the stock is \$18 if the calls are exercised?
(d) What will be the investor's profit at expiration if the spot price of the stock is \$16 and the investor sells his shares?

Problem 64.9
An investor sells short 200 shares of ABC stock at \$5.25 a share. He sells two put contracts (100 shares each) with a striking price of \$5 and a premium of \$0.50 a share.
(a) What is the investor's profit if the spot price is below \$50 at expiration?
(b) The investor's will be making profit as long as the spot price is less than what value at expiration?

65 Synthetic Forward and Put-Call Parity

A **synthetic forward** is a combination of a long call and a short put with the same expiration date and strike price.

Example 65.1
An investor buys a 500-strike call option with expiration date in three months, risk-free 3-month interest of 1% and a premium of \$46.90. He also sells a 500-strike put option with expiration date in three months, risk-free 3-month interest of 1% and a premium of \$41.95.
(a) Show that the investor is obliged to buy the index for \$500 at expiration date.
(b) Draw the profit diagram of the long call, the short put, and the combined position of the long call and the written put.
(c) State two differences between the synthetic long forward and the actual long forward.

Solution.
(a) If the spot price at expiration is greater than \$500, the put will not be exercised (and thus expires worthless) but the investor will exercise the call. So the investor will buy the index for \$500. If the spot price at expiration is smaller than \$500, the call will not be exercised and the investor will be assigned on the short put, if the owner of the put wishes to sell then the investor is obliged to buy the index for \$500. Either way, the investor is obliged to buy the index for \$500 and the long call-short put combination induces a long forward contract that is synthetic since it was fabricated from options.
(b) The payoff of the combined position is $\max\{0, \text{spot price} - \text{strike price}\} - \max\{0, \text{strike price} - \text{spot}\} = \text{spot price} - \text{strike price}$. But this is the payoff of a long forward contract expiring at time T and with forward price of \$500.
The profit for the combined position is $\max\{0, \text{spot} - \text{strike}\} - 46.90(1.01) - \max\{0, \text{strike} - \text{spot}\} + 41.95(1.01)$. The profit diagram of each position is given in Figure 65.1.

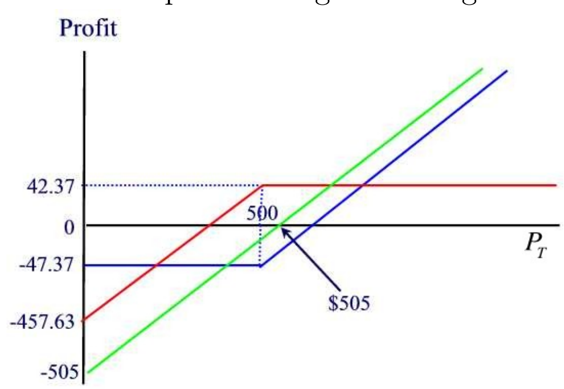


Figure 65.1

Thus, a synthetic long forward is equivalent to a long forward with forward price of \$505 (= 500(1.01)).
(c) The actual forward (i.e. the long forward with forward price \$505) premium is zero whereas there is a net option premium of $46.90 - 41.95 = \$4.95$ with the synthetic forward. At expiration, we pay the forward price (i.e. \$505) with the actual forward whereas with the synthetic forward we pay the strike price of the options (i.e. \$500) ■

Let $F_{0,T}$ denote the no-arbitrage (i.e. zero cost) forward price . This means we pay \$0 today and at time T we are obliged to buy the asset at the price $F_{0,T}$. The cost of the contract today is the present value of $F_{0,T}$ which we will denote by $PV(F_{0,T})$. Note that $PV(F_{0,T}) = S_0$, where S_0 is the current asset price. Let Call(K,T) and Put(K,T) denote the premiums of the purchased call and the written put respectively. We pay Call$(K,T) -$ Put(K,T) today to buy the assets for K at time T which gives a cost at time 0 of Call$(K,T) -$ Put$(K,T) + PV(K)$.
Using the no-arbitrage pricing, the net cost of asset must be the same whether through options or forward contract, that is,

$$\text{Call}(K,T) - \text{Put}(K,T) + PV(K) = PV(F_{0,T})$$

or

$$\text{Call}(K,T) - \text{Put}(K,T) = PV(F_{0,T} - K) \tag{65.2}$$

Equation (65.2) is known as **put-call parity**. This is an important relationship; it "ties" the option and forward markets together. Note that if the strike price of a synthetic forward equals the no-arbitrage forward price, then the net option premium is zero. In this case, the premium paid for the purchased call equals the premium received for the written put.
Note that a synthetic long forward is a forward contract with a premium as opposed to an actual forward contract which does not require a premium. A forward contract with a premium is called an **off-market forward**. Thus, unless the strike price and the no-arbitrage forward price are equal, buying a call and selling a put creates an off market forward.

Example 65.2
The premium of a 6-month off-market (synthetic) forward contract with a forward price of \$1000 is \$19.61. The premium of a 6-month 1000-strike call is \$X and that of a 6-month 1000-strike put is \$74.20, determine X.

Solution.
Using Equation (65.2) we obtain

$$X - 74.20 = 19.61.$$

Solving for X we find $X = \$93.81$ ■

Equation (65.2) can be rearranged to show the equivalence of the prices (and payoffs and profits) of a variety of different combinations of positions.

Example 65.3
Show that buying an index plus a put option with strike price K is equivalent to buying a call option with strike price K and a zero-coupon bond with par value of K.

Solution.
The cost of the index is $PV(F_{0,T})$ and that of the put option is Put(K, T). The cost of the call option is Call(K, T) and that of the zero-coupon bond is $PV(K)$. From Equation (65.2) we can write

$$PV(F_{0,T}) + \text{Put(K, T)} = \text{Call(K, T)} + PV(K)$$

which establishes the equivalence of the two positions ■

Example 65.4
Show that shorting an index plus buying a call option with strike price K is equivalent to buying a put option with strike price K and taking out a loan with maturity value of K.

Solution.
The cost of the short index is $-PV(F_{0,T})$ and that of the call option is Call(K, T). The cost of the put option is Put(K, T) and the loan is $-PV(K)$. From Equation (65.2) we can write

$$-PV(F_{0,T}) + \text{Call(K, T)} = \text{Put(K, T)} - PV(K)$$

which establishes the equivalence of the two positions ■

Remark 65.1
Recall that neither a zero-coupon bond nor borrowing affect the profit function.

Example 65.5
A call option on XYZ stock with an exercise price of \$75 and an expiration date one year from now is worth \$5.00 today. A put option on XYZ stock with an exercise price of \$75 and an expiration date one year from now is worth \$2.75 today. The annual risk-free rate of return is 8% and XYZ stock pays no dividends. Find the current price of the stock.

Solution.
Using the Put-Call parity we have

$$PV(F_{0,T}) = \text{Call(K, T)} - \text{Put(K, T)} + 75(1.08)^{-1} = \$71.69 \blacksquare$$

Example 65.6
A short synthetic forward contract can be created by reversing the long synthetic forward. This is done by combining a purchased put with a written call. Let K be the strike price and T the expiration time on both options. Show that the payoff for the short synthetic forward contract is given by $K - P_T$.

Solution.
The payoff of the short synthetic forward is the payoff of the purchased put plus the payoff of the written call which is given by the expression $\max\{K - P_T, 0\} - \max\{K - P_t, 0\}$. Whether $P_T \geq K$ or $P_T < K$ one can easily check that the payoff is $K - P_T$ ■

Practice Problems

Problem 65.1
If a synthetic forward contract has no initial premium then
(A) The premium you pay for the call is larger than the premium you receive from the put
(B) The premium you pay for the call is smaller than the premium you receive from the put
(C) The premium you pay for the call is equal to the premium you receive from the put
(D) None of the above.

Problem 65.2
In words, the Put-Call parity equation says that (A) The cost of buying the asset using options must equal the cost of buying the asset using a forward.
(B) The cost of buying the asset using options must be greater than the cost of buying the asset using a forward.
(C) The cost of buying the asset using options must be smaller than the cost of buying the asset using a forward.
(D) None of the above.

Problem 65.3
According to the put-call parity, the payoffs associated with ownership of a call option can be replicated by
(A) shorting the underlying stock, borrowing the present value of the exercise price, and writing a put on the same underlying stock and with same exercise price
(B) buying the underlying stock, borrowing the present value of the exercise price, and buying a put on the same underlying stock and with same exercise price
(C) buying the underlying stock, borrowing the present value of the exercise price, and writing a put on the same underlying stock and with same exercise price
(D) None of the above

Problem 65.4
State two features that differentiate a synthetic forward contract from a no-arbitrage forward contract.

Problem 65.5
Show that buying an index plus selling a call option with strike price K (i.e. selling a covered call) is equivalent to selling a put option with strike price K and buying a zero-coupon bond with par value K.

Problem 65.6
Show that short selling an index plus selling a put option with strike price K (i.e. selling a covered put) is equivalent to selling a call option with strike price K and taking out a loan with maturity value of K.

Problem 65.7 ‡
You are given the following information:
• The current price to buy one share of XYZ stock is 500.
• The stock does not pay dividends.
• The risk-free interest rate, compounded continuously, is 6%.
• A European call option on one share of XYZ stock with a strike price of K that expires in one year costs \$66.59.
• A European put option on one share of XYZ stock with a strike price of K that expires in one year costs \$18.64.
Using put-call parity, determine the strike price, K.

Problem 65.8
The current price of an stock is \$1000 and the stock pays no dividends in the coming year. The premium for a one-year European call is \$93.809 and the premium for the corresponding put is \$74.201. The annual risk-free interest rate is 4%. Determine the strike price of the synthetic forward.

Problem 65.9
The actual forward price of a stock is \$1020 and the stock pays no dividends in the coming year. The premium for a one-year European call is \$93.809 and the premium for the corresponding put is \$74.201. The strike price of the synthetic forward contract is \$1000. Determine the annual risk-free effective rate r.

Problem 65.10
Describe how you can financially engineer the payoff to holding a stock by a combination of a put, call and a zero-coupon bond. Assume that the put and call have the same maturity date T and exercise price K, and that the bond has also has maturity date T and face value K.

Problem 65.11
A stock currently sell for \$100 and the stock pays no dividends. You buy a call with strike price \$105 that expires in one year for a premium \$3.46. You sell a put with the same expiration date and the strike price. Assuming, a risk-free effective annual rate of 6%, determine the put's premium.

Problem 65.12
A \$50, one-year call sells for \$4; a \$50 one-year put sells for \$6.45. If the one-year interest rate is 8%, calculate the implied stock price.

Problem 65.13
XYZ Corporation sells for \$35 per share; the AUG option series has exactly six months until expiration. At the moment, the AUG 35 call sells for \$3, and the AUG 35 put sells for \$1.375. Using this information, what annual interest rate is implied in the prices?

Problem 65.14 ‡
You are given the following information:
• One share of the PS index currently sells for 1,000.
• The PS index does not pay dividends.
• The effective annual risk-free interest rate is 5%.
You want to lock in the ability to buy this index in one year for a price of 1,025. You can do this by buying or selling European put and call options with a strike price of 1,025. Which of the following will achieve your objective and also gives the cost today of establishing this position.
(A) Buy the put and sell the call, receive 23.81
(B) Buy the put and sell the call, spend 23.81
(C) Buy the put and sell the call, no cost
(D) Buy the call and sell the put, receive 23.81
(E) Buy the call and sell the put, spend 23.81

66 Spread Strategies

Forming a **spread strategy** with options means creating a position consisting only calls or only puts, in which some options are purchased and some are sold. In this section, we discuss some of the typical spread strategies.

Bull Spreads
A position that consists of buying a call with strike price K_1 and expiration T and selling a call with strike price $K_2 > K_1$ and same expiration date is called a **bull call spread.** In contrast, buying a put with strike price K_1 and expiration T and selling a put with strike price $K_2 > K_1$ and same expiration date is called a **bull put spread.** An investor who enters a bull spread is speculating that the stock price will increase.

Example 66.1
Find a formula of the payoff of a bull call spread.

Solution.
The payoff for buying a K_1-strike call with expiration date T is $\max\{0, P_T - K_1\}$. The payoff for selling a K_2-strike call (where $K_2 > K_1$) with the same expiration date is $-\max\{0, P_T - K_2\}$. Thus, the payoff for the call bull spread is $\max\{0, P_T - K_1\} - \max\{0, P_T - K_2\}$. Now, if $P_T \leq K_1$ then the call bull payoff is 0. If $K_1 < P_T < K_2$ then the call bull payoff is $P_T - K_1$. Finally, if $K_2 \leq P_T$ then the call bull payoff is $K_2 - K_1$. The payoff diagram for the bull call spread is given in Figure 66.1 ■

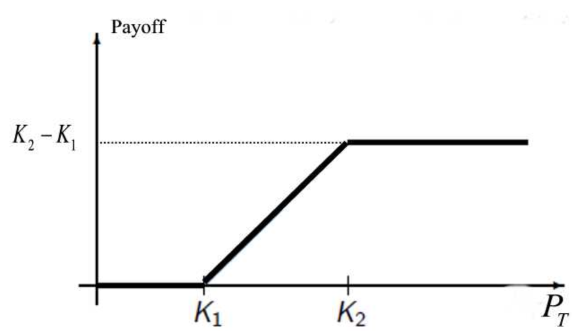


Figure 66.1

Example 66.2
Find a formula of the profit of a bull call spread.

Solution.
The profit of the position is the profit of the long call plus the profit of the short call. Thus,

the profit is the payoff of the combined position minus the future value of the net premium. If $P_T \leq K_1$ then the profit is $-FV(\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T}))$. If $K_1 < P_T < K_2$ then the profit is $P_T - K_1 - FV(\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T}))$. If $K_2 \leq P_T$ then the profit is $K_2 - K_1 - FV(\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T}))$. Figure 66.2 shows a graph of the profit of a bull call spread. Note that from the graph, a bull spread limits the investor's risk but also limits the profit potential ■

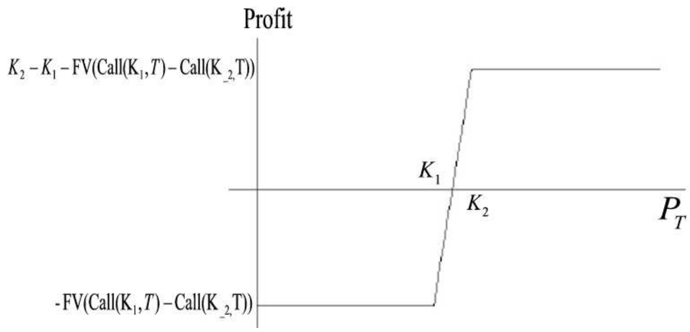


Figure 66.2

Example 66.3
An investor buys a \$70-strike call and sells a \$85-strike call of a stock. Both have expiration date one year from now. The risk free annual effective rate of interest is 5%. The premiums of the \$70-strike and \$85-strike calls are \$10.76 and \$3.68 respectively.
(a) Find the profit at expiration as a function of the spot price.
(b) Complete the following table

Stock Price at Expiration	Total Profit
65	
70	
75	
80	
85	
90	

(c) Draw the profit diagram.

Solution.
(a) The future value of the premium is $(10.76 - 3.68)(1.05) = 7.43$. The profit function as a function of P_T is given by

$$\left\{\begin{array}{cc} -7.43 & \text{if } P_T \leq 70 \\ P_T - 70 - 7.43 & \text{if } 70 < P_T \leq < 85 \\ 85 - 70 - 7.43 & \text{if } 85 \leq P_T \end{array}\right.$$

(b)

Stock Price at Expiration	Total Profit
65	−7.43
70	−7.43
75	−2.43
80	2.57
85	7.57
90	7.57

(c) The profit diagram is similar Figure 66.2 ■

Example 66.4
Show that the profit diagram of bull call spread coincides with the profit diagram of the bull put spread.

Solution.
As a function of P_T the profit from buying a K_1-strike call and selling a K_2-strike call with the same expiration date is given by

$$\begin{cases} -FV(\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T})) & \text{if } P_T \leq K_1 \\ P_T - K_1 - FV(\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T})) & \text{if } K_1 < P_T \leq< K_2 \\ K_2 - K_1 - FV(\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T})) & \text{if } K_2 \leq P_T \end{cases}$$

The profit from buying a K_1-strike put with expiration date T and premium $\text{Put}(\text{K}_1, \text{T})$ is $\max\{0, K_1 - P_T\} - FV(\text{Put}(\text{K}_1, \text{T}))$. The profit from selling a K_2-strike put (where $K_2 > K_1$) with the same expiration date is $-\max\{0, K_2 - P_T\} + FV(\text{Put}(\text{K}_2, \text{T}))$. Thus, the profit for the put bull spread is $\max\{0, K_1 - P_T\} - \max\{0, K_2 - P_T\} - FV(\text{Put}(\text{K}_1, \text{T}) - \text{Put}(\text{K}_2, \text{T}))$. The profit as a funtion of P_T is given by

$$\begin{cases} K_1 - K_2 - FV(\text{Put}(\text{K}_1, \text{T}) - \text{Put}(\text{K}_2, \text{T})) & \text{if } P_T \leq K_1 \\ P_T - K_2 - FV(\text{Put}(\text{K}_1, \text{T}) - \text{Put}(\text{K}_2, \text{T})) & \text{if } K_1 < P_T \leq< K_2 \\ -FV(\text{Put}(\text{K}_1, \text{T}) - \text{Put}(\text{K}_2, \text{T})) & \text{if } K_2 \leq P_T \end{cases}$$

Using the Put-Call parity we obtain the equation

$$FV([\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T})] - [\text{Put}(\text{K}_1, \text{T}) - \text{Put}(\text{K}_2, \text{T})]) = K_2 - K_1.$$

This equation shows that the two profits coincide. Hence, we can form a bull spread either buying a K_1-strike call and selling a K_2-strike call, or buying a K_1-strike put and selling a K_2-strike put ■

Bear Spread
A **bear spread is precisely** the opposite of a bull spread. An investor who enters a bull spread is hoping that the stock price will increase. By contrast, an investor who enters a bear spread is hoping that the stock price will decline. Let $0 < K_1 < K_2$. A bear spread can be created by either selling a K_1-strike call and buying a K_2-strike call, both with the same expiration date (bear call spread), or by selling a K_1-strike put and buying a K_2-strike put, both with the same expiration date (bear put spread).

Example 66.5
Find a formula of the profit of a bear spread created by selling a K_1-strike call and buying a K_2-strike call, both with the same expiration date. Graph the profit diagram.

Solution.
The profit of the position is the profit of the short call plus the profit of the long call. Thus, the profit is the payoff of the combined position minus the future value of the net premium. That is, $\max\{0, P_T - K_2\} - \max\{0, P_T - K_1\} + FV(\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T}))$. The profit function is given by

$$\begin{cases} FV(\text{Call}(\text{K}_1,\text{T}) - \text{Call}(\text{K}_2,\text{T})) & \text{if } P_T \leq K_1 \\ K_1 - P_T + FV(\text{Call}(\text{K}_1,\text{T}) - \text{Call}(\text{K}_2,\text{T})) & \text{if } K_1 < P_T \leq < K_2 \\ K_1 - K_2 + FV(\text{Call}(\text{K}_1,\text{T}) - \text{Call}(\text{K}_2,\text{T})) & \text{if } K_2 \leq P_T \end{cases}$$

Figure 66.3 shows a graph of the profit of a bear spread. Note that the graph is the reflection of the graph in Figure 66.2 about the horizontal axis ■

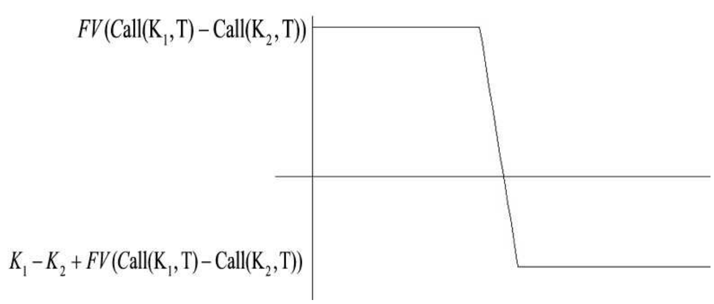


Figure 66.3

Combination of a Bull Spread and a Bear Spread: Box Spread
A **long box spread** comprises four options, on the same underlying asset with the same expiration date. They can be paired in two ways as shown in the following table

	Bull Call Spread	Bear Put Spread
Synthetic Long Forward	Buy Call at K_1	Sell Put at K_1
Synthetic Short Forward	Sell Call at K_2	Buy Put at K_2

Reading the table horizontally and vertically we obtain two views of a long box spread.
• A long box spread can be viewed as a long synthetic stock at a price plus a short synthetic stock at a higher price.
• A long box spread can be viewed as a long bull call spread at one pair of strike prices plus a long bear put spread at the same pair of strike prices.
No matter what the spot price of the stock is at expiration, the box spread strategy guarantees a cash flow of $K_2 - K_1$ in the future. Thus, this strategy has no stock price risk.
Using Put-Call parity (assuming no-arbitrage), the net premium of acquiring this position is given by

$$(\mathrm{Call}(K_1, T) - \mathrm{Put}(K_1, T)) - (\mathrm{Call}(K_2, T) - \mathrm{Put}(K_2, T)) = PV(K_2 - K_1).$$

If $K_1 < K_2$, a box spread is a way to lend money. An investment of $PV(K_2 - K_1)$ is made at time zero and a return of $K_2 - K_1$ per share is obtained at time T.
If $K_1 > K_2$, a box spread is a way to borrow money. A return of $PV(K_1 - K_2)$ is received at time zero and a loan payment of $K_1 - K_2$ is made at time T.

Example 66.6
Find an expression for the payoff at expiration of a (long) box spread.

Solution.
The payoff is the payoff of a synthetic long forward $(P_T - K_1)$ plus the payoff of a synthetic short forward $(K_2 - P_T)$. That is, $K_2 - K_1$. Hence, the payoff at expiration is just the difference between the strike prices of the options involved. The profit will be the amount by which the discounted payoff exceeds the net premium. Under no-arbitrage, the profit is zero. Normally, the discounted payoff would differ little from the net premium, and any nominal profit would be consumed by transaction costs. ■

Example 66.7
Consider a three-month option on a stock whose current price is \$40. If the effective annual risk-free interest rate is 8.33% then price for the options might be

K	Call	Put
40	2.78	1.99
45	0.97	5.08

Find the cost at time zero and the payoff at expiration of the box spread obtained by
• buying a 40-strike call and selling a 40-strike put
• selling a 45-strike call and buying a 45-strike put

Solution.
The cost at time zero is the discounted value of the payoff $45 - 40 = 5$ which is $5 \times (1.0833)^{0.25} =$ \$4.90 ■

Ratio Spreads
A **(call) ratio spread** is achieved by buying a certain number of calls with a low strike and selling a different number of calls at a higher strike. By replacing the calls with puts one gets a **(put) ratio spread.** All options under considerations have the same expiration date and same underlying asset. If m calls were bought and n calls were sold we say that the ratio is $\frac{m}{n}$.

Example 66.8
An investor buys one \$70-strike call and sells two \$85-strike call of a stock. All the calls have expiration date one year from now. The risk free annual effective rate of interest is 5%. The premiums of the \$70-strike and \$85-strike calls are \$10.76 and \$3.68 respectively. Draw the profit diagram of this position.

Solution.
The profit of the 1:2 ratio is given by

$$\max\{0, \text{spot} - 70\} - 2\max\{0, \text{spot} - 85\} + (2 \times 3.68 - 10.76) \times 1.05.$$

The profit diagram is given in Figure 66.4

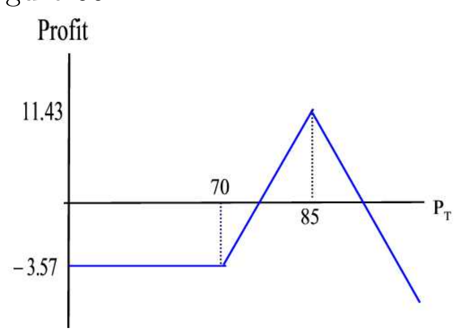


Figure 66.4

Practice Problems

Problem 66.1
Explain two ways in which a bull spread can be created.

Problem 66.2
The premium of a 3-month 40-strike call option on a stock currently selling for \$40 is \$2.78, and the premium for a 45-strike call option on the same stock is \$0.97. Consider the bull spread achieved by purchasing the 40-strike call and selling the 45-strike call. The risk-free effective annual interest rate is 8.33%.
(a) Find the future value of the net premium.
(b) Graph the profit function of the bull spread.

Problem 66.3
The premium of a 3-month 40-strike put option on a stock currently selling for \$40 is \$1.99, and the premium for a 45-strike put option on the same stock is \$5.08. Consider the bull spread achieved by purchasing the 40-strike put and selling the 45-strike put. The risk-free effective annual interest rate is 8.33%.
(a) Find the future value of the net premium.
(b) Graph the profit function of the bull spread.

Problem 66.4
A 3-month call options with strike prices \$35 and \$40 have premiums \$6.13 and \$2.78, respectively. The risk-free effective annual interest rate is 8.33%.
(a) What is the maximum gain when a bull spread is created from calls?
(b) What is the maximum loss when a bull spread is created from calls?

Problem 66.5
True or false:For a bull put spread I would sell a put option with a low strike price, and buy a put option with a higher strike price.

Problem 66.6
Explain two ways in which a bear spread can be created.

Problem 66.7
A 3-month call options with strike prices \$35 and \$40 have premiums \$6.13 and \$2.78, respectively. The risk-free effective annual interest rate is 8.33%.
(a) What is the maximum gain when a bear spread is created from calls?
(b) What is the maximum loss when a bear spread is created from calls?

Problem 66.8
Suppose you buy a 100-strike call, sell a 120-strike call, sell a 100-strike put, and buy a 120-strike put. Assume effective annual risk-free interest rate of 8.5% and the expiration date of the options is one year.
(a) Verify that there is no price risk in this transaction.
(b) What is the initial cost of the position?
(c) What is the value of the position at expiration?

Problem 66.9
An investor buys two \$70-strike call and sells one \$85-strike call of a stock. All the calls have expiration date one year from now. The risk free annual effective rate of interest is 5%. The premiums of the \$70-strike and \$85-strike calls are \$10.76 and \$3.68 respectively. Draw the profit diagram of this position.

Problem 66.10
Which of the following will create a bull spread
(A) Buy a put with strike price \$50, sell a put with strike price 55
(B) Buy a put with strike price \$55, sell a put with strike price 50
(C) Buy a call with premium of \$5, sell a call with premium \$7
(D) Buy a call with strike price \$50, sell a put with strike \$55.

67 Collars

A **collar** is achieved with the purchase of an at-the-money put option with strike price K_1 and expiration date T and the selling of an out-of-the-money call option with strike price $K_2 > K_1$ and expiration date T. Both options use the same underlying asset. A collar can be used to speculate on the decrease of price of an asset. The difference $K_2 - K_1$ is called the **collar width**.
A **written collar** is the reverse of collar (sale of a put and purchase of a call).

Example 67.1
Find the profit function of a collar as a function of P_T.

Solution.
The profit function is

$$\max\{0, K_1 - P_T\} - \max\{0, P_T - K_2\} + FV(\text{Call}(\text{K}_2, \text{T}) - \text{Put}(\text{K}_1, \text{T}))$$

which is

$$\begin{cases} K_1 - P_T + FV(\text{Call}(\text{K}_2, \text{T}) - \text{Put}(\text{K}_1, \text{T})) & \text{for } P_T \le K_1 \\ FV(\text{Call}(\text{K}_2, \text{T}) - \text{Put}(\text{K}_1, \text{T})) & \text{for } K_1 < P_T < K_2 \\ K_2 - P_T + FV(\text{Call}(\text{K}_2, \text{T}) - \text{Put}(\text{K}_1, \text{T})) & \text{for } K_2 \le P_T \end{cases} \blacksquare$$

Example 67.2
An investor buys a 65-strike put with premium \$1.22 and sells an 80-strike call with premium \$5.44. Both have expiration date one year from now. The current price of the stock is \$65. The risk free annual effective rate of interest is 5%. Draw the profit diagram of this collar.

Solution.
Using the function from the previous example we can create the following table

P_T	55	60	65	70	75	80	85	90
Profit	14.43	9.43	4.43	4.43	4.43	4.43	−0.57	−5.57

The profit diagram is shown in Figure 67.1

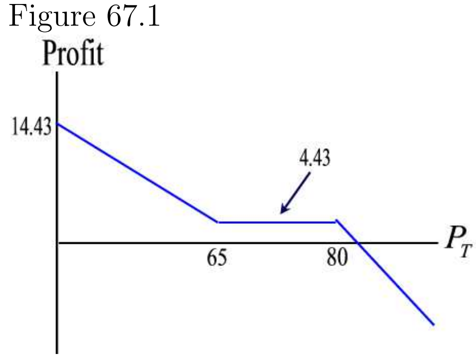


Figure 67.1

Note that the figure depicts a short position. The position benefits from declining price asset and suffers losses from price asset appreciation. It resembles a short forward with the only exception that with a collar there a range between the strikes in which the profit is unaffected by changes in the value of the underlying asset ■

Collars can be used to insure assets we own. This position is called a **collared stock.** A collared stock involves buying the index, buy an at-the-money K_1-strike put option (which insures the index) and selling an out-of-the-money K_2-strike call option (to reduce the cost of the insurance), where $K_1 < K_2$.

Example 67.3
Find the profit function of a collared stock as a function of P_T.

Solution.
The profit function at expiration is

$$P_T + \max\{0, K_1 - P_T\} - \max\{0, P_T - K_2\} + FV(\mathrm{Call(K_2, T)} - \mathrm{Put(K_1, T)} - P_0)$$

which is

$$\begin{cases} K_1 + FV(\mathrm{Call(K_2, T)} - \mathrm{Put(K_1, T)} - P_0) & \text{for } P_T \le K_1 \\ P_T + FV(\mathrm{Call(K_2, T)} - \mathrm{Put(K_1, T)} - P_0) & \text{for } K_1 < P_T < K_2 \\ K_2 + FV(\mathrm{Call(K_2, T)} - \mathrm{Put(K_1, T)} - P_0) & \text{for } K_2 \le P_T \blacksquare \end{cases}$$

Example 67.4
Amin buys an index for \$65. To insure the index he buys a 65-strike put with premium \$1.22. In order to reduce the cost of this insurance he sells an 80-strike call with premium \$5.44. Both options have expiration date one year from now. The risk free annual effective rate of interest is 5%. Draw the profit diagram of this collar.

Solution.
Using the function from the previous example we can create the following table

P_T	55	60	65	70	75	80	85	90
Profit	1.181	1.181	1.181	6.18	11.18	16.18	16.18	16.18

The profit diagram is shown in Figure 67.2

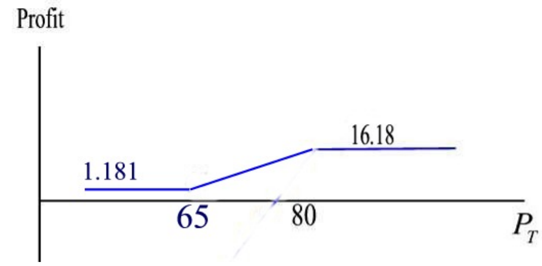


Figure 67.2

Note that the profit diagram resembles the one of a bull spread ■

Zero-cost Collars
A collar with zero cost at time 0, i.e. with zero net premium, is called a **zero-cost collar** (also known as a **costless collar**). In this strategy, one buys an at-the-money put and selling an out-of-the-money call with both having the same premium.

Example 67.5
Suppose that a zero-cost collar consists of buying a K_1-strike put option and selling a K_2-strike call option, where $K_1 < K_2$. Find the profit function as a function of P_T.

Solution.
The profit coincides with the payoff since the net premium is zero at time 0. Hence, the cost function is

$$\max\{0, K_1 - P_T\} - \max\{0, P_T - K_2\}$$

which is

$$\begin{cases} K_1 - P_T & \text{for } P_T \le K_1 \\ 0 & \text{for } K_1 < P_T < K_2 \\ K_2 - P_T & \text{for } K_2 \le P_T \end{cases}$$ ■

Example 67.6
A zero-cost collar on an index is created by buying the index for \$60, buying a 55-strike put and selling a 65-strike call. Both options have expiration date one year from now. The risk free annual effective rate of interest is 5%. Draw the profit diagram of this position.

Solution.
The profit function is

$$\begin{cases} 55 - 60(1.05) & \text{for } P_T \leq 55 \\ P_T - 60(1.05) & \text{for } 55 < P_T < 65 \\ 65 - 60(1.05) & \text{for } 65 \leq P_T \end{cases} \blacksquare$$

We have the following table

P_T	45	50	55	60	65	70	75
Profit	-8	-8	-8	-3	2	2	2

The profit diagram is shown in Figure 67.3

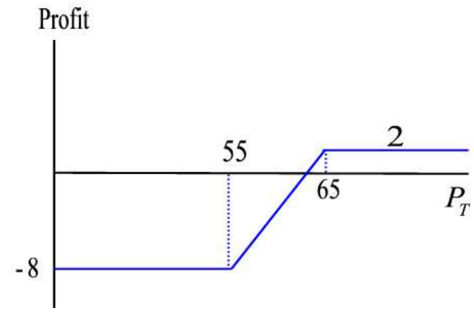


Figure 67.3

Remark 67.1
(1) For a given stock, it can be shown that there are an infinite number of possible combinations of strike prices that will produce a zero-cost collar. That is, there is an infinite number of zero-cost collars.
(2) If $K_1 = K_2 = F_{0,T}$ (i.e. you will receive $F_{0,T}$ at time T) then by the Put-Call parity the cost of the collar is zero and the collar is identical to a forward contract (a short forward in the case of a purchased collar (See Figure 67.1), or a long forward in the case of a written collar (See Problem 67.6).
(c) In (b), it sounds that you are insuring your asset for free. That is not the case. In fact, you have lost money because you have forgone interest on the asset.

Practice Problems

Problem 67.1
A purchased collar consists of a 40-strike put and a 65-strike call. What is the collar width?

Problem 67.2
A purchased collar is achieved by
(A) Buying a K_1-strike put and selling a K_2-strike call with $K_1 < K_2$
(B) Selling a K_1-strike put and buying a K_2-strike call with $K_1 < K_2$
(A) Buying a K_1-strike call and selling a K_2-strike put with $K_1 < K_2$
(A) Selling a K_1-strike call and buying a K_2-strike put with $K_1 < K_2$

Problem 67.3
A written collar is achieved by
(A) Buying a K_1-strike put and selling a K_2-strike call with $K_1 < K_2$
(B) Selling a K_1-strike put and buying a K_2-strike call with $K_1 < K_2$
(A) Buying a K_1-strike call and selling a K_2-strike put with $K_1 < K_2$
(A) Selling a K_1-strike call and buying a K_2-strike put with $K_1 < K_2$

Problem 67.4 ‡
Which statement about zero-cost purchased collars is FALSE?
(A) A zero-width, zero-cost collar can be created by setting both the put and call strike prices at the forward price.
(B) There are an infinite number of zero-cost collars.
(C) The put option can be at-the-money.
(D) The call option can be at-the-money.
(E) The strike price on the put option must be at or below the forward price.

Problem 67.5
Find the profit function of a written collar as a function of P_T.

Problem 67.6
An investor sells a 65-strike put with premium \$1.22 and buys an 80-strike call with premium \$5.44. Both have expiration date one year from now. The current price of the stock is \$65. The risk free annual effective rate of interest is 5%. Draw the profit diagram of this written collar.

Problem 67.7
To insure a short sale of a stock, an investor buys a K_1-strike call option (which insures the index)

and sells a K_2-strike put option (to reduce the cost of the insurance), where $K_1 > K_2$. Find the profit function of this position as a function of P_T. What spread does the diagram resemble to?

Problem 67.8
Amin short sale an index for \$75. To insure the index he buys a 80-strike call with premium \$5.44. In order to reduce the cost of this insurance he sells a 65-strike put with premium \$1.22. Both options have expiration date one year from now. The risk free annual effective rate of interest is 5%. Draw the profit diagram of this collar. What spread strategy does the diagram resemble to?

Problem 67.9 ‡
Happy Jalapenos, LLC has an exclusive contract to supply jalapeno peppers to the organizers of the annual jalapeno eating contest. The contract states that the contest organizers will take delivery of 10,000 jalapenos in one year at the market price. It will cost Happy Jalapenos 1,000 to provide 10,000 jalapenos and today's market price is 0.12 for one jalapeno. The continuously compounded risk-free interest rate is 6%.
Happy Jalapenos has decided to hedge as follows (both options are one-year, European):
Buy 10,000 0.12-strike put options for 84.30 and sell 10,000 0.14-strike call options for 74.80.
Happy Jalapenos believes the market price in one year will be somewhere between 0.10 and 0.15 per pepper. Which interval represents the range of possible profit one year from now for Happy Jalapenos?
(A) −200 to 100
(B) −110 to 190
(C) −100 to 200
(D) 190 to 390
(E) 200 to 400

68 Volatility Speculation: Straddles, Strangles, and Butterfly Spreads

In this section we consider strategies that depend on the volatility of a financial market rather than on the direction of the asset price change. For example, straddles (to be defined below) create positive payoffs when the market price is way different from the options strike price in either direction (i.e. up or down). That is, the investor with these strategies does not care whether the stock price goes up or down, but only how much it moves.
The first strategy that we examine is that of a straddle. A **long straddle** or simply a **straddle** is an option strategy that is achieved by buying a $K-$strike call and a $K-$strike put with the same expiration time T and same underlying asset. This strategy has a high premium since it requires purchasing two options. However, there is a guaranteed positive payoff with this strategy as long as the stock price at expiration is different from the strike price (whether higher or lower). Indeed, the investor benefits from the call if the stock price appreciates and from the put if the stock price declines. If the spot price at expiration is almost equal to the strike price the premiums from the two options are lost.
The owner of a straddle is betting on high volatile market, regardless of the direction of price movement (up or down). Premiums of straddles become greater when the market's perception is that volatility is greater.
The payoff of a straddle is the sum of the payoff of a purchased call and a purchased put. That is,

$$\max\{0, K - P_T\} + \max\{0, P_T - K\} = |P_T - K|.$$

The profit function is given by

$$|P_T - K| - FV(\text{Call(K, T)} + \text{Put(K, T)}).$$

The profit diagram is shown in Figure 68.1. Note that a straddle has a limited risk, since the most a purchaser may lose is the cost of both options. At the same time, there is unlimited profit potential.

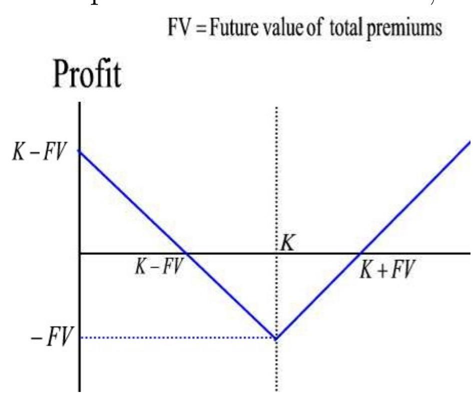


Figure 68.1

Example 68.1
Tess buys a 40-strike call option and a 40-strike put of an index. Both have expiration date three months from now. The current price of the index is \$40. The risk free annual effective rate of interest is 8.33%. The premium of the call option is \$2.78 and that of the put option is \$1.99.
(a) Determine the future value of the initial cost.
(b) Find the profit as a function of the spot price at expiration P_T.
(c) Find the values of the spot price at expiration at which Tess makes a profit.

Solution.
(a) The future value of the initial cost is $(2.78 + 1.99)(1.0833)^{0.25} = \$4.87$.
(b) The profit is given by

$$|P_T - 40| - 4.87.$$

(c) From Figure 68.1, a profit occurs when $0 < P_T < 35.13$ or $P_T > 44.87$ ■

As pointed out above, a straddle has a high premium cost. One way to reduce the cost is to buy out-of-the-money options. Such a strategy is called a **strangle**. More formally, a strangle is a strategy that invovles buying an out-of-the-money call and an out-of-the-money put option with different strike prices but with the same maturity and underlying asset. For example, buy a K_1-strike put and a K_2-strike call with the same expiration date and same underlying asset and such that $K_1 < K_2$. The profit of such a strangle is

$$\max\{0, K_1 - P_T\} + \max\{0, P_T - K_2\} - FV(\text{Put}(\text{K}_1, \text{T}) + \text{Call}(\text{K}_2, \text{T}))$$

or

$$\begin{cases} K_1 - P_T - FV(\text{Put}(\text{K}_1, \text{T}) + \text{Call}(\text{K}_2, \text{T})) & \text{if } P_T \leq K_1 \\ -FV(\text{Put}(\text{K}_1, \text{T}) + \text{Call}(\text{K}_2, \text{T})) & \text{if } K_1 < P_T < K_2 \\ P_T - K_2 - FV(\text{Put}(\text{K}_1, \text{T}) + \text{Call}(\text{K}_2, \text{T})) & \text{if } K_2 \leq P_T. \end{cases}$$

In Poblem 68.10 we consider the profit of a strangle with $K_2 < K_1$.

Example 68.2
Drew buys a 35-strike put option and a 45-strike call of an index. Both have expiration date three months from now. The current price of the index is \$40. The risk free annual effective rate of interest is 8.33%. The premium of the call option is \$0.97 and that of the put option is \$0.44.
(a) Determine the future value of the initial cost.
(b) Find the profit as a function of the spot price at expiration P_T.
(c) Find the break-even prices.
(d) Draw the profit diagram of this position.
(e) On the same window, draw the profit diagram of the straddle position of Example 68.1.

Solution.
(a) The future value of the initial cost is $(0.97 + 0.44)(1.0833)^{0.25} = \$1.44$.
(b) The profit is given by

$$\begin{cases} 35 - P_T - 1.44 & \text{if } P_T \leq 35 \\ -1.44 & \text{if } 35 < P_T < 45 \\ P_T - 45 - 1.44 & \text{if } 45 \leq P_T. \end{cases}$$

(c) The break-even prices occur at $P_T = 33.56$ and $P_T = 46.44$
(d) The profit diagram is shown in Figure 68.2

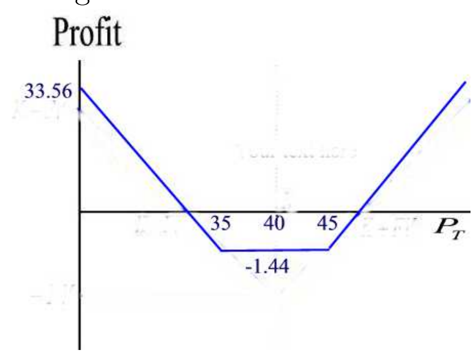


Figure 68.2

(e) The profit diagram of the straddle as well as the one for the strangle are shown in Figure 68.3 ■

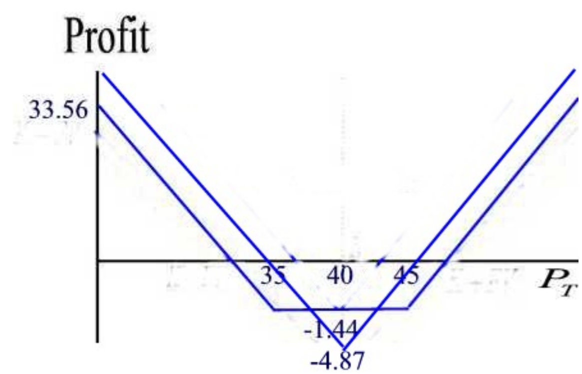


Figure 68.3

Some observations are in order. Even though these options reduce the initial cost (and therefore the maximum loss), they increase the stock-price move required to make a profit. The profit diagrams of the two positions interesect at the points where $-P_T+40-4.87 = -1.44$ and $P_T-40-4.87 = -1.44$. That is, at the points $P_T = 36.57$ and $P_T = 43.43$. Only, in the interval $36.57 < P_T < 43.43$, the strangle outperforms the straddle.

A straddle is a bet on high volatility. A corresponding position that bets on low volatility is a **written straddle**. This position is achieved by selling a $K-$strike call and a $K-$strike put with the same expiration date and same underlying asset. Similarly, a **written strangle** is a bet on low volatility. It is achieved by selling a K_1-strike call and a K_2-strike put ($K_1 < K_2$) with the same expiration date and same underlying asset.
The profit of a written straddle is given by

$$-|P_T - K| + FV(\text{Call(K, T)} + \text{Put(K, T)}).$$

The profit diagram is shown in Figure 68.4. Note that a written straddle is most profitable if the stock price is K at expiration, and in this sense it is a bet on low volatility. However, a large change in the direction of the stock price results in a potentially unlimited loss.

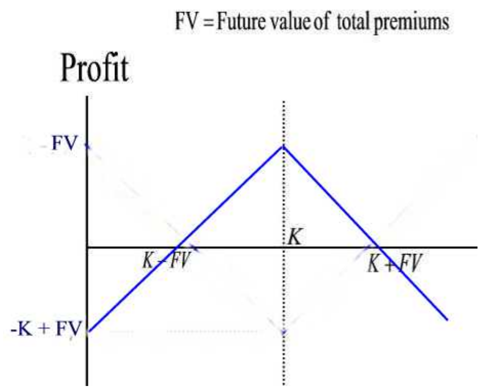


Figure 68.4

One way to limit the losses that can occur with a written straddle is by buying options to insure against losses whether is stock price is falling or rising. Buying an out-of-the-money put protects against losses that can occur by a sharp decline in the at-the-money written put. Buying an out-of-the-money call protects against losses that can occur by a sharp increase in the at-the-money written call. An insured written straddle strategy is referred to as a **butterfly spread.**
Suppose that $K_1 < K_2 < K_3$. One way to create a butterfly spread is by using the following combination:
(1) Create a written straddle by selling a K_2-strike call and a K_2-strike put.
(2) Create a long strangle by buying a K_1-strike call and a K_3-strike put.
The purchased strangle provides insurance against the written straddle. All the options have the same expiration date T and the same underlying asset.
The initial cost of the position is

$$\text{Call}(\text{K}_1, \text{T}) - \text{Call}(\text{K}_2, \text{T}) - \text{Put}(\text{K}_2, \text{T}) + \text{Put}(\text{K}_3, \text{T}).$$

Letting FV denote the future value of the net premium, the profit is given by

$$\max\{0, P_T - K_1\} + \max\{0, K_3 - P_T\} - \max\{0, P_T - K_2\} - \max\{0, K_2 - P_T\} - FV$$

or

$$\begin{cases} K_3 - K_2 - FV & \text{if } P_T \leq K_1 \\ P_T - K_1 - K_2 + K_3 - FV & \text{if } K_1 < P_T < K_2 \\ -P_T - K_1 + K_2 + K_3 - FV & \text{if } K_2 \leq P_T < K_3 \\ K_2 - K_1 - FV & \text{if } K_3 \leq P_T. \end{cases}$$

Example 68.3
Sophia sells a 40-strike call option (with premium \$2.78) and a 40-strike put (with premium \$1.99) of an index. To insure against unlimited losses she buys a 35strike call option (with premium \$6.13) and a 45strike put (with premium \$5.08). All the options have expiration date three months from now. The current price of the index is \$40. The risk free annual effective rate of interest is 8.33%. Draw the profit diagram of this position.

Solution.
The future value of the net premium is $(6.13 - 2.78 - 1.99 + 5.08)(1.0833)^{0.25} = \$6.57$. The profit function is given by

$$\begin{cases} 45 - 40 - 6.57 & \text{if } P_T \leq 35 \\ P_T - 35 - 40 + 45 - 6.57 & \text{if } 35 < P_T < 40 \\ -P_T - 35 + 40 + 45 - 6.57 & \text{if } 40 \leq P_T < 45 \\ 40 - 35 - 6.57 & \text{if } 45 \leq P_T. \end{cases}$$

The profit diagram is shown in Figure 68.5 ■

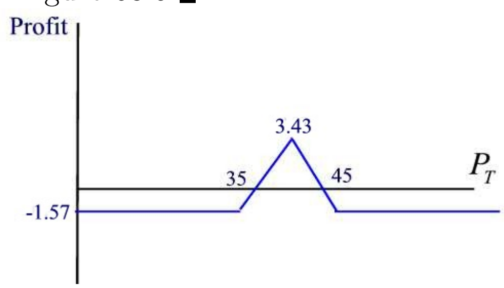


Figure 68.5

A butterfly spread may be created by using only calls, only puts, or by a long (or a short) position combined with both calls and puts. See Problems 68.3 - Problems 68.5.

Given $0 < K_1 < K_2 < K_3$. When a symmetric butterfly is created using these strike prices the number K_2 is the midpoint of the interval with endpoints K_1 and K_3. What if K_2 is not midaway between K_1 and K_3? In this case, one can create a butterfly-like spread with the peak tilted either to the left or to the right as follows: Define a number λ by the formula

$$\lambda = \frac{K_3 - K_2}{K_3 - K_1}.$$

Then λ satisfies the equation

$$K_2 = \lambda K_1 + (1 - \lambda)K_3.$$

Thus, for every written K_2-strike call, a butterfly-like spread can be constructed by buying λ K_1-strike calls and $(1-\lambda)$ K_3-strike calls. The resulting spread is an example of an **asymmetric butterfly spread.**

Example 68.4
Construct an an asymmetric butterfly spread using the 35-strike call (with premium \$6.13), 43-strike call (with premium \$1.525) and 45-strike call (with premium \$0.97). All options expire 3 months from now. The risk free annual effective rate of interest is 8.33%. Draw the profit diagram of this position.

Solution.
With the given strike prices we find Then $\lambda = 0.2$. Thus, an asymmetric butterfly spread is created by selling ten 43-strike calls, buying two 35-strike calls and eight 45-strike calls. The future value of the net premium is $(2 \times 6.13 + 8 \times 0.97 - 10 \times 1.525)(1.0833)^{0.25} = \$4.87$. The profit is

$$2\max\{0, P_T - 35\} + 8\max\{0, P_T - 45\} - 10\max\{0, P_T - 43\} - 4.87.$$

or

$$\begin{cases} -4.87 & \text{if } P_T \le 35 \\ 2P_T - 74.87 & \text{if } 35 < P_T < 43 \\ -8P_T + 355.13 & \text{if } 43 \le P_T < 45 \\ -4.87 & \text{if } 45 \le P_T. \end{cases}$$

The profit diagram is shown in Figure 68.6 ■

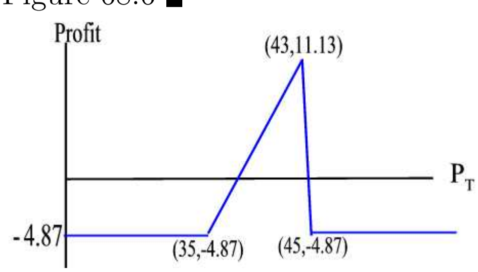


Figure 68.6

As with the symmetric butterfly spread, an investor can create an asymmetric butterfly spread by using only calls, using only puts, or by using a long or short position in the stock, combined with both puts and calls. See Problem 68.12.

Practice Problems

Problem 68.1
Which of the following is most equivalent to writing a straddle?
(A) buy stock, write two calls
(B) buy stock, buy one put
(C) short stock, buy one call
(D) short stock, buy one put.

Problem 68.2
A butterfly spread of an index is created as follows:
(1) buying a 35-strike call with premium \$6.13 and selling a 40-strike call with premium \$2.78 (that is buying a bull call spread).
(2) selling a 40-strike call with premium \$2.78 and buyin a 45-strike call with premium \$0.97 (that is buying a bear call spread).
All options expire in three months from now. The current price of the index is \$40. The risk free annual effective rate of interest is 8.33%. Show that the profit diagram of this position coincides with the one in Example 68.3.

Problem 68.3
A butterfly spread of an index is created by buying a 35-strike call (with premium \$6.13), selling two 40-strike calls (with premium \$2.78 each) and buying a 45-strike call (with premium \$0.97).
All options expire in three months from now. The current price of the index is \$40. The risk free annual effective rate of interest is 8.33%. Show that the profit diagram of this position coincides with the one in Example 68.3.

Problem 68.4
A butterfly spread of an index is created by buying a 35-strike put (with premium \$0.44), selling two 40-strike puts (with premium \$1.99 each) and buying a 45-strike put (with premium \$5.08).
All options expire in three months from now. The current price of the index is \$40. The risk free annual effective rate of interest is 8.33%. Show that the profit diagram of this position coincides with the one in Example 68.3.

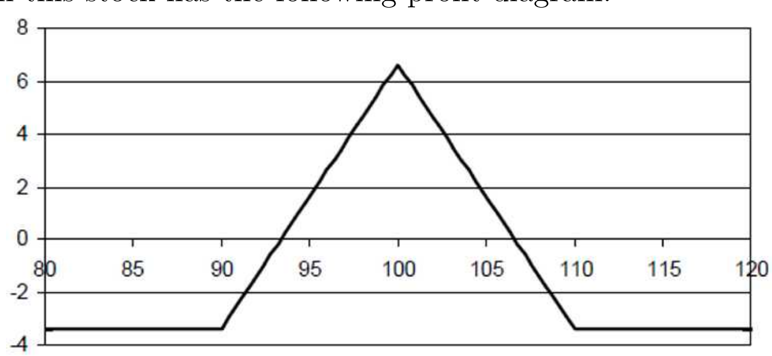
Problem 68.5
A butterfly spread of an index is created by buying the stock, buying a 35-strike put (with premium \$0.44), selling two 40-strike calls (with premium \$2.78 each) and buying a 45-strike call (with premium \$0.97).
All options expire in three months from now. The current price of the index is \$40. The risk free annual effective rate of interest is 8.33%. Show that the profit diagram of this position coincides with the one in Example 68.3.

Problem 68.6 ‡
You believe that the volatility of a stock is higher than indicated by market prices for options on that stock. You want to speculate on that belief by buying or selling at-the-money options. What should you do?
(A) Buy a strangle
(B) Buy a straddle
(C) Sell a straddle
(D) Buy a butterfly spread
(E) Sell a butterfly spread

Problem 68.7 ‡
You are given the following information:
• The current price to buy one share of ABC stock is 100
• The stock does not pay dividends
• The risk-free rate, compounded continuously, is 5%
• European options on one share of ABC stock expiring in one year have the following prices:

Strike Price	Call Premium	Put Premium
90	14.63	0.24
100	6.80	1.93
110	2.17	6.81

A butterfly spread on this stock has the following profit diagram.



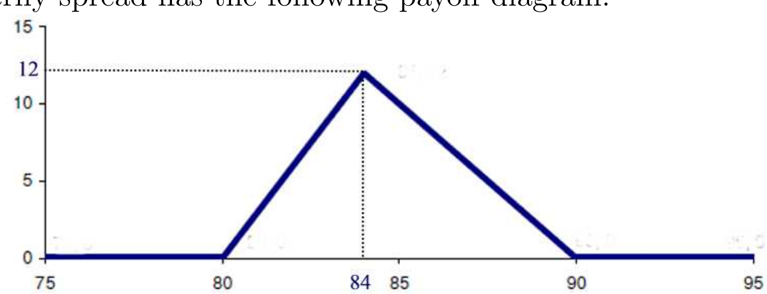
Which of the following will NOT produce this profit diagram?
(A) Buy a 90 put, buy a 110 put, sell two 100 puts
(B) Buy a 90 call, buy a 110 call, sell two 100 calls
(C) Buy a 90 put, sell a 100 put, sell a 100 call, buy a 110 call
(D) Buy one share of the stock, buy a 90 call, buy a 110 put, sell two 100 puts
(E) Buy one share of the stock, buy a 90 put, buy a 110 call, sell two 100 calls.

Problem 68.8
Which of the following has the potential to lose the most money?
(A) a long butterfly spread
(B) a short strangle
(C) a long strangle
(D) an asymetric butterfly spread

Problem 68.9
Three month European put options with strike prices of \$50, \$55, and \$60 cost \$2, \$4, and \$7, respectively. A butterfly is created by buying a 50-strike put, selling two 55-strike put, and buying a 60-strike put.
The risk-free 3-month interest rate is 5%.
(a) What is the maximum gain when a butterfly spread is created from the put options?
(b) What is the maximum loss when a butterfly spread is created from the put options?
(c) For what two values of P_T does the holder of the butterfly spread breakeven, where P_T is the stock price in three months?

Problem 68.10
Consider a strangle with the strike price of the put is higher than the strike price of the call. Draw the profit diagram.

Problem 68.11
An asymmetric butterfly spread has the following payoff diagram:



This position was created using calls that are priced as follows:

Strike Price	Premium
80	4
84	2
90	0.50

Ignoring commissions and bid-ask spreads, what was the cost to establish this asymmetric butterfly spread?

Problem 68.12

Show that the asymmetric butterfly of Figure 68.6 can be duplicated by buying two 35-strike puts (with premium \$0.44 each), selling ten 43-strike puts (with premium \$3.674 each) and buying eight 45-strike puts (with premium \$5.08 each). All options expire 3 months from now. The risk free annual effective rate of interest is 8.33%.

69 Equity Linked CDs

An **equity-linked CD** (ELCD) (also known as **index-linked CD**) is an FDIC-insured certificate of deposit that ties the rate of return to the performance of the market indices such as the S&P 500 Composite Stock Price Index, the Dow Jones Industrial Average, or the NASDAQ 100 composite index. Equity-linked CDs are usually issued by a bank.
The CDs may vary by maturity (five years, six years, etc.), participation rate (some offer 100% of the increase in the market index, others only 90% or 70%), and the method of computing the return of the stock index (some use an averaging method to determine the ending value of the index level, while others use the final index value). For example, an equity-linked CD can have the following structure: A CD with a return linked to the S&P 500 index is guaranteed to repay the invested amount plus 70% of the simple appreciation in the S&P 500 at the maturity date of 5.5 years. If the invested amount is $10,000 and the index is 1300 initially then a decline in the index after 5.5 years will result in a return of the $10,000 to the investor. If on the other hand the index rises to 2200 at the maturity date then the return to the investor is the original investment plus 70% of the precentage gain on the index. That is,

$$10,000\left[1+0.70\left(\frac{2200}{1300}-1\right)\right]=\$14,846.$$

More generally, the payoff is given by the formula

$$10,000\left[1+0.70\max\left\{0,\frac{S_{5.5}}{1300}-1\right\}\right]$$

where $S_{5.5}$ is the value of the index after 5.5 years.

Example 69.1
A 5.5-year ELCD promises to repay $10,000 and 70% of the gain in S&P 500 index at expiration date. Assume investment occurs when S&P 500 index was 1300.
(a) Complete the following table

Index after 5.5 years	Payoff
500	
1000	
1500	
2000	
2500	
3000	

(b) Draw the payoff diagram.

Solution.
(a)

Index after 5.5 years	Payoff
500	10,000
1000	10,000
1500	11,076.92
2000	13,769.23
2500	16,461.54
3000	19,153.85

(b) The payoff diagram is shown in Figure 69.1

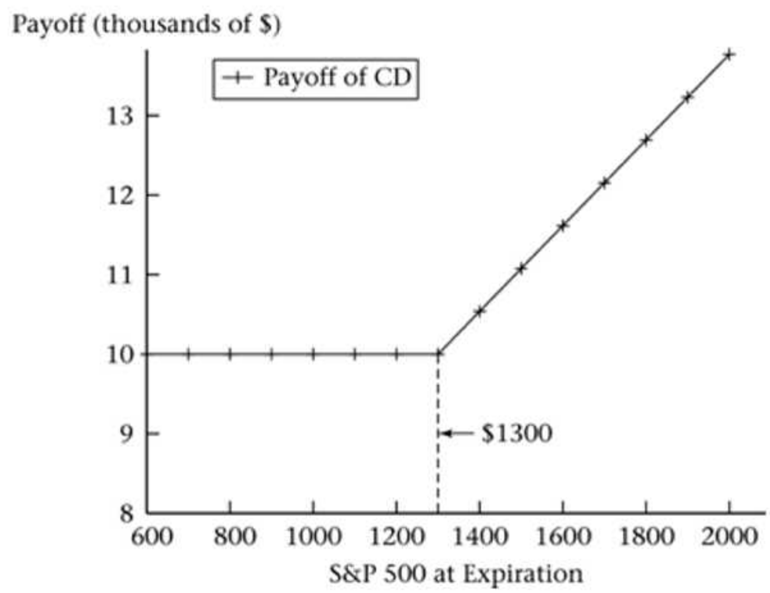


Figure 69.1

Building an ELCD Using a Zero-Coupon Bond and a Call Option
We next use financial engineering to create the building structure of an ELCD by buying a zero-coupon bond and a call option.
Consider an investor who wants to invest $1,000 in a S&P 500 index with no risk of losing principal and the opportunity to earn market-like returns. This investor might purchase a zero-coupon bond and an at-the-money call option on the S&P 500 market index. The investor buys the zero-coupon

bond at a discount and receives the \$1,000 bond par value at maturity. She then uses the difference between the \$1,000 investment and the price of the zero-coupon bond to pay for the call option premium. The call option has expiration date as the time to maturity of the bond. If, at maturity, the S&P 500 has increased, the investor exercises the call option and earns a return on the capital gains portion of the S&P 500. If the S&P 500 decreases, the investor does not exercise the option and receives nothing for it. However, the investor still receives the face value of the bond thereby insuring the original \$1,000 investment.
A setback of the ELCD in the above example is that in the case of a decline in the index the investor will get back his \$1,000 at maturity date. If, say, the maturity date is 5 years and if the annual effective interest rate during the investment time is 6%, the investor has lost in interest

$$1,000(1.06)^5 - 1,000 = \$338.22.$$

The present value of this interest is $338.22(1.06)^{-5} = \$252.74$. Thus as the investor invests today he has foregone interest with a present value of \$252.74, and that is the implied cost today of investing in the CD.
Summarizing, an investor is attracted to ELCD's because it has the potential for market appreciation without risking capital. One disadvantage is the possible loss of interest on the invested principal.
In general, the payoff of an ELCD with invested principal P and rate of participation r is given by the formula

$$P\left(1 + r\max\{\frac{P_T}{P_0} - 1, 0\}\right) = P + Pr\max\{\frac{P_T}{P_0} - 1, 0\}.$$

That is, the payoff of a typical equity-linked CD at expiration is equal to the par amount of the CD plus an equity-linked coupon. In general, the equity-linked coupon is equal to either:
(a) zero, if the underlying equity has depreciated from an agreed upon strike level (usually the index level on the issue date of the CD), or
(b) the participation rate times the percentage change in the underlying equity times the par amount of the note, if the underlying equity appreciated.

Example 69.2
An investor buys a five-year ELCD with 100% participation in the simple appreciation of the S&P 500 index for \$1,000. The starting index-level is 1,400.
(a) Find the payoff if after five years the index-level is 2,100.
(b) Find the payoff if after five years the index-level is below 1,400.
(c) Create the payoff diagram.

Solution.
(a) The payoff is

$$1,000\left[1 + \left(\frac{2100}{1400} - 1\right) \times 100\%\right] = \$1,500.$$

(b) The total payoff is just the original investment of \$,1000.
(c) The profit diagram is shown in Figure 69.2 ■

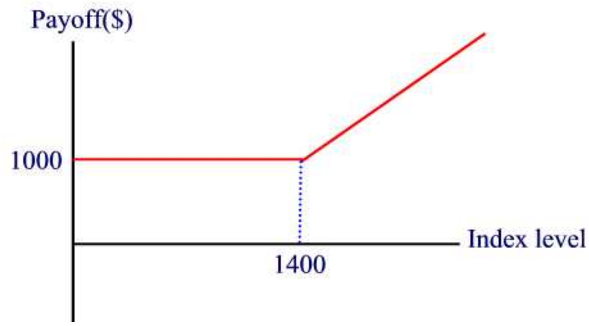


Figure 69.2

Example 69.3
Create a synthetic equivalent to the ELCD in Example 69.2.

Solution.
A synthetic equivalent can be created as follows:
(1) invest in a five-year 6% discount bond with par value of \$1,000 for $1,000(1.06)^{-5} = \$747.26$ and
(2) buy a five-year, S&P 500 at-the-money call option with a \$1,400 strike price for a premium \$252.74 ■

We conclude this section by looking at how the three perspectives the end-user, the market-maker, and the economic observer view an ELCD:
• The end-user is interested in the product and whether it meets a financial need at a fair cost.
• The market-maker (i.e. the bank issuer) is interested in making profit without bearing risk from having issued the CD. Some of the profit come from fees for issuing the CD and from penalities of early withdrawals since ELCDs cannot be withdrawn before maturity date.
• The economic observer studies the advantage of having ELCDs.

Practice Problems

Problem 69.1
The payoff in Example 69.1 can be regarded as the payoff of $10,000 and the payoff of a certain unit of European call options. What is that number of units?

Problem 69.2
Suppose that an ELCD promises to repay the principal investment plus 80% of the gain of a linked equity. If the gain on the equity is 50% during the investment period then what is the gain for each dollar invested?

Problem 69.3
In a ELCD, an investor will receive a return consisting of his orginal investment only when
(a) the underlying index has increased from the starting index value
(b) the underlying index has remained static or have fallen.

Problem 69.4
An investor has an equity linked CD with a nominal value of $10,000 and a participation of %110. If the positive movement of the equity markets at maturity is 35% then find the amount due at maturity to the investor.

Problem 69.5
Which of the following statements are true about equity linked CDs:
(I) pays a fixed interest rate
(II) return depends on the performance of the underlying assets
(III) Miniumum return is the initially invested money
(IV) can be viewed as a combination of a zero-coupon bond and a call option

Problem 69.6
A 5-year CD promises to repay $1,000,000 and 115% of the gain in S&P 500 index at expiration date. Assume investment occurs when S&P 500 index was 500.
(a) Complete the following table

Index after 5 years	Payoff
400	
450	
500	
550	
600	
700	
800	

(b) Draw the payoff diagram.

Problem 69.7
What is the minimum value of a call option? An equity-linked CD?

Problem 69.8
A 5-year CD promises to repay $1,000,000 and 115% of the gain in S&P 500 index at expiration date. Assume investment occurs when S&P 500 index was 500. Assume annual effective interest rate of 8% during the 5-year period. What is the implied cost today of investing in the equity-linked index ?

Problem 69.9
A 5-year CD promises to repay $1,000,000 and 115% of the gain in S&P 500 index at expiration date. Assume investment occurs when S&P 500 index was 500. Create a synthetic ELCD for this investment.

70 Prepaid Forward Contracts On Stock

There are four alternative ways for buying shares of a stock. Namely,
• **Outright purchase:** An investor simultaneously pays S_0 dollars in cash at time 0 and owns the stock.
• **Full leveraged purchase:** An investor borrows S_0 dollars and receives the ownership of the stock at time 0. Letting r denote the annual continuous compound interest, the investor has to repay S_0e^{rt} at time T.
• **Forward contract:** An arrangement for an agreed-upon future price (the forward price) and date in which the buyer pays for the stock at time T and the seller transfers the ownership of the stock to the buyer at that time.
• **Prepaid forward contract or prepay:** This is like a forward contract but with the (prepaid) forward price paid at time 0. That is, the contract entails the buyer to pay today for the stock and owns it in the future. The price one pays is not necessarily the stock price S_0.
In this section, we will discuss the question of pricing a prepaid forward contract.

Pricing of Prepaid Forward Contracts with no Dividends
Let $F_{0,T}^P$ denote the prepaid forward price for an asset bought at time 0 and delivered at time T. This price depends on the dividends paid to the stockholders. In the absence of dividends, the timing of delivery is irrelevant. Therefore, the prepaid forward price is just the stock price at time 0. That is,

$$F_{0,T}^P = S_0.$$

Example 70.1
The current price of a stock is \$50. The free-risk interest rate is 4% compounded continuously. Assuming no dividends, what is the prepaid forward price?

Solution.
The price of a prepaid forward contract on a stock that pays no dividends is the initial stock price. So the answer is \$50 ■

Now, arbitrage opportunities (positive cash flows either today or in the future and with no risk) can be achieved with prepaid forward price as illustrated in the following two examples.

Example 70.2
Suppose that the prepaid forward price exceeds the stock price. An arbitrageur will buy the stock at the low price of S_0 and sell the prepaid forward contract for $F_{0,T}^P$. Assuming no dividends, show that this transaction has a positive cash flow at time 0 and is risk-free.

Solution.
The net cash flow at time 0 is $F^P_{0,T} - S_0 > 0$ since the cash flow (outflow) from buying the stock is $-S_0$ and from selling the prepaid contract (inflow) is $F^P_{0,T}$. This transaction is risk-free: Selling the prepaid forward contract obliges the seller to deliver the stock at time T and buying the stock today ensures that the stock will be delivered at that time ■

Example 70.3
Suppose that the stock price exceeds the prepaid forward. An arbitrageur will short the stock at the price of S_0 and buy the prepaid forward contract for $F^P_{0,T}$. Assuming no dividends, show that this transaction has a positive cash flow at time 0 and is risk-free.

Solution.
The net cash flow at time 0 is $S_0 - F^P_{0,T} > 0$ since the cash flow (outflow) from buying the prepaid forward is $-F_{0,T}$ and from selling the stock (inflow) is S_0. This transaction is risk-free: By buying the prepaid forward contract we are entitled to acquire the stock at time T which we use to close the short position ■

Throughout the book all prices are assumed to be free arbitrage prices.

Pricing of Prepaid Forward Contracts with Dividends
Stocks that pay dividends have prepaid forward price lower than the stock price. Only the owner of the stock receives the dividends but not the holder of the prepaid forward.
When dividend is paid the stock price is reduced by the amount of the dividend. Suppose that over the life of a forward contract, a stock receives dividends D_i at time t_i where $i = 1, 2, \cdots, n$. Then the prepaid forward price is lower than the stock price by the present value of those dividends:

$$F^P_{0,T} = S_0 - \sum_{i=1}^{n} PV(D_i). \tag{70.1}$$

Example 70.4
Suppose ABC stock costs \$75 today and is expected to pay semi-annual dividend of \$1 with the first coming in 4 months from today and the last just prior to the delivery of the stock. Suppose that the annual continuously compounded risk-free rate is 8%. Find the cost of a 1-year prepaid forward contract.

Solution.
The cost is

$$F^P_{0,1} = 75 - e^{-0.08\times\frac{4}{12}} - e^{-0.08\times\frac{10}{12}} = \$73.09 \blacksquare$$

Formula (70.1) applies for the case of discrete dividends. What about a stock index where dividends are given almost daily? In this case, it is common to model the dividends as being continuously paid at a constant **continuoulsy compounded dividend yield rate** δ (which is defined to be the annualized dividend payment divided by the stock price). Note that δ is used here as a continuously compounded dividend rate, rather than a force of interest. Like an interest rate, the dividend rate is a rate *per year.*
Divide the interval $[0, T]$ into n equal subintervals using the mesh point $t_i = \frac{i}{n}T$ where $1 \leq i \leq n$. The dollar dividend over each interval is

$$\text{Dividend} = \frac{\delta T}{n} S_0.$$

Suppose that dividends payments are reinvested into stock. Let A_i be the total number of shares at time t_i. Then $A_{i+1} = A_i \left(1 + \frac{\delta T}{n}\right)$. Hence, the total amount of shares multiplies by $\left(1 + \frac{\delta T}{n}\right)$ so that one share at time 0 grows to $\left(1 + \frac{\delta T}{n}\right)^n$ at time T. Letting $n \to \infty$ we get that one share at time 0 grows to $e^{\delta T}$ shares at time T.

Example 70.5
You buy one share of ABC stock and hold it for two years. The dividends are paid at continuously compounded rate of 3% and you reinvest all the dividends when they are paid. The current stock price is \$125.
(a) Find the daily dollar dividend.
(b) How many shares do you have at the end of two years?

Solution.
(a) The daily dollar dividend is

$$\text{Dividend} = \frac{0.03}{365} \times 125 = \$0.01027.$$

(b) At the end of two years, the number of shares increased from one share to $e^{0.03 \times 2} = 1.061836$ shares ■

Now, if one share of the index is desired at time T, then one has to buy $e^{-\delta T}$ shares today. Due to dividend reinvestment, at time T we will have

$$e^{-\delta T} e^{\delta T} = 1 \text{ share}.$$

This process of buying just enough shares today so that when dividends are reinvested we accumulate exactly one share at time T is called **tailing** the position. The cost per share at time 0 of a tailed position is $e^{-\delta T} S_0$.
This position can be achieved as well by buying a $T-$year prepaid forward contract on a stock (or

index) that pays dividends at a continuous rate of δ for delivery of one full share of the index at time T. If there exists no arbitrage, the prepaid forward price must be equal to the cost of the tailing position at time 0. That is,

$$F_{0,T}^P = S_0 e^{-\delta T}.$$

Remark 70.1
In either the case of a tailed position or a prepaid forward contratct, the buyer of the share does not receive (in cash) any dividends paid prior to time T. In the case of the tailed position, those dividends are used to buy additional stock to bring the total to one full share; in the case of the prepaid forward contract, dividends are not payable to the holder of a long forward prior to the expiration (delivery) date, time T (at which time the investor receives one full share).

Example 70.6
Find the 1-year prepaid forward price of the index in the previous example.

Solution.
The prepaid forward price is

$$F_{0,1}^P = 125e^{-0.03} = \$121.306 \blacksquare$$

Practice Problems

Problem 70.1
Complete the following charts that descibe the cash flows and transactions to undertake arbitrage (stocks pay no dividends).

$F_{0,T} > S_0$

	Cash Flows	
	$t = 0$	$t = T$
Buy Stock		
Sell Prepaid Forward		
Total		

$F_{0,T} < S_0$

	Cash Flows	
	$t = 0$	$t = T$
Short Stock		
Buy Prepaid Forward		
Total		

Problem 70.2
Suppose that the current price of a stock is \$1100. The stock does not pay dividends and the risk-free annual rate is 5%.
(a) Suppose that an investor is willing to buy a prepaid forward contract at a prepaid forward price of \$1135 on a one year prepaid forward contract. Show how to make an arbitrage gain under these circumstances.
(b) Suppose that an investor is willing to sell a prepaid forward contract at a prepaid forward price of \$1050 on a one year prepaid forward contract. Show how to make an arbitrage gain under these circumstances.

Problem 70.3
Suppose ABC stock costs \$X today. It is expected that 4 quarterly dividends of \$1.25 each will be paid on the stock with the first coming 3 months from now. The 4th dividend will be paid one day before expiration of the forward contract. Suppose the annual continuously compounded risk-free rate is 10%. Find X if the cost of the forward contract is \$95.30. Round your answer to the nearest dollar.

Problem 70.4
Suppose ABC stock costs \$75 today and is expected to pay semi-annual dividend of \$X with the

first coming in 4 months from today and the last just prior to the delivery of the stock. Suppose that the annual continuously compounded risk-free rate is 8%. Find X if the cost of a 1-year prepaid forward contract is \$73.09. Round your answer to the nearest dollar.

Problem 70.5
You buy one share of Ford stock and hold it for 2 years. The dividends are paid at the annualized daily continuously compounded rate of 3.98% and you reinvest all the dividends when they are paid. How many shares do you have at the end of two years?

Problem 70.6
Suppose that annual dividend of 30 on the stocks of an index is valued at \$1500. What is the continuously compounded dividend yield?

Problem 70.7
Suppose XYZ stock costs \$50 today and is expected to pay quarterly divident of \$1 with the first coming in 3 months from today and the last just prior to the delivery of the stock. Suppose that the annual continuously compounded risk-free rate is 6%. What is the price of a prepaid forward contract that expires 1 year from today, immediately after the fourth-quarter dividend?

Problem 70.8
Suppose XYZ stock costs \$50 today and is expected to pay 8% continuous risk-free dividend. What is the price of a prepaid forward contract that expires 1 year from today, immediately after the fourth-quarter dividend?

Problem 70.9
LEPOs (Low Exercised Price Options) exist in order to avoid taxes and transaction fees associated with trading with stocks. These stocks have zero-dividends and are purchased outright. These options are certain to be exercised due to the fact that the strike price K is so low. The payoff at expiration time T is $P_T - K$. Find the cost of this option at time 0. Show that a LEPO is essentially a prepaid forward contract with no dividends.

Problem 70.10
An investor is interested in buying XYZ stock. The current price of stock is \$45 per share. This stock pays dividends at an annual continuous risk-free rate of 5%. Calculate the price of a prepaid forward contract which expires in 18 months.

Problem 70.11 ‡
A non-dividend paying stock currently sells for 100. One year from now the stock sells for 110. The risk-free rate, compounded continuously, is 6%. The stock is purchased in the following manner:

• You pay 100 today
• You take possession of the security in one year
Which of the following describes this arrangement?

(A) Outright purchase
(B) Fully leveraged purchase
(C) Prepaid forward contract
(D) Forward contract
(E) This arrangement is not possible due to arbitrage opportunities

71 Forward Contracts on Stock

Recall that prepaid forward contracts are forward contracts with payment made at time 0. The forward price is the future value of the prepaid forward price and this is true regardless of whether there are discrete dividends, continuous dividends, or no dividends
If the expiration time is T and the continuous risk-free rate is r then for a stock with no dividends (i.e. $F_{0,T}^P = S_0$) the future value of the prepaid forward price is given by the formula

$$F_{0,T} = S_0 e^{rT}.$$

In the case of a stock with discrete dividends $D_1, D_2, \cdots, D_n$ made at time $t_1, t_2, \cdots, t_n$ the forward price is

$$\begin{aligned} F_{0,T} =& FV(F_{0,T}^P) = \left(S_0 - \sum_{i=1}^n D_i e^{-rt_i}\right) e^{rT} \\ =& S_0 e^{rT} - \sum_{i=1}^n D_i e^{r(T-t_i)} \end{aligned}$$

In the case of a stock with continuous dividends with continuously compounded yield δ we have

$$F_{0,T} = (S_0 e^{-\delta T}) e^{rT} = S_0 e^{(r-\delta)T}.$$

The **forward premium** is the ratio of the forward price to the current spot price defined by

$$\text{Forward premium} = \frac{F_{0,T}}{S_0}.$$

The expression

$$1 - \frac{F_{0,T}}{S_0}$$

is the percentage by which the forward price exceeds (or falls below) the spot price.
A number α that satisfies the equation

$$F_{0,T} = S_0 e^{\alpha T}$$

is called the **annualized forward premium.** Solving this equation for α we find

$$\alpha = \frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0}\right).$$

If a stock pays no dividends, the annualized forward premium is just the annual risk-free interest rate $\alpha = r$.
For the case of continuous dividends, the annualized forward premium reduces to

$$\alpha = \frac{1}{T}\ln\left(\frac{S_0e^{(r-\delta)T}}{S_0}\right) = r - \delta.$$

Remark 71.1
In the literature, the forward premium of an exchange rate forward contract is defined to be the percentage by which today's forward rate exceeds (falls below) today's spot rate and is given by the formula

$$\ln\left(\frac{F_{0,T}}{S_0}\right).$$

Usually these premia are quoted as annualized rates given by the formula

$$\frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right).$$

Example 71.1
Suppose the stock price is \$90 and the continuously compounded interest rate is 10%.
(a) What is the 9-month forward price assuming dividends are zero? What is the forward premium?
(b) If the 9-month forward price is \$92, what is the annualized forward premium?
(c) If the 9-month forward price is \$92, what is the annualized continuous dividend yield?

Solution.
(a) $F_{0,\frac{9}{12}} = 90e^{0.1\times\frac{9}{12}} = \$97.01$. The forward premium is $\frac{97.01}{90} = 1.07789$.
(b) The annualized forward premium is

$$\alpha = \frac{1}{T}\ln\left(\frac{F_{0,9/12}}{S_0}\right) = \frac{12}{9}\ln\left(\frac{92}{90}\right) = 0.0293.$$

(c) We have $0.0293 = 0.10 - \delta$. Solving for δ we find $\delta = 0.0707 = 7.07\%$ ■

In Section 65, we have seen how to create a synthetic forward contract by buying a call and selling a put with both having the same strike price. We next discuss another way for creating a synthetic forward.
In this discussion we consider a stock that pays continuous dividends at the rate δ. We buy a tailed position by buying $e^{-\delta T}$ shares of the stock which cost $S_0e^{-\delta T}$ at time 0 so that at time T we have one share of the stock. We borrow $S_0e^{-\delta T}$ to pay for the shares of stock. In this case, we invest 0 at time 0. At time T we must repay $S_0e^{(r-\delta)T}$ and we sell the one share of stock at price P_T. The payoff for this position is

$$\text{Payoff at expiration} = P_T - S_0e^{(r-\delta)T} = P_T - F_{0,T}$$

which is the payoff of a long forward contract with forward price $F_{0,T}$ and expiration time T. In terms of a word equation we have

$$\text{Forward} = \text{Stock} - \text{Zero coupon bond} \tag{71.1}$$

By rearranging the terms of this equation we can derive a **synthetic stock**

$$\text{Stock} = \text{Forward} + \text{Zero-coupon bond}$$

This position is created by buying a forward contract with forward price $F_{0,T} = S_0e^{(r-\delta)T}$ and expiration T and lending $S_0e^{-\delta T}$ (the present value of the forward price) at time 0. The payoff at time T is

$$\text{Payoff at expiration} = (P_T - F_{0,T}) + F_{0,T} = P_T$$

which is the payoff of buying $e^{-\delta T}$ shares of the stock. We can use (71.1) to create a **synthetic bond**

$$\text{Zero-coupon bond} = \text{Stock} - \text{Forward}.$$

This position is created by buying $e^{-\delta T}$ shares of the index at the cost of $S_0e^{-\delta T}$ at time 0 and short one forward contract with forward price $F_{0,T}$ and expiration time T. The payoff of this position at time T is

$$\text{Payoff at expiration} = P_T + (F_{0,T} - P_T) = F_{0,T}$$

which is the payoff of a long risk-free bond with par value of $F_{0,T}$ and expiration time T.

Example 71.2

Consider the position of selling $e^{-\delta T}$ shares for $S_0e^{-\delta T}$ at time 0 and buying a forward with forward price $F_{0,T} = S_0e^{(r-\delta)T}$. Complete the following table

	Cash Flows	
	$t = 0$	$t = T$
Selling $e^{-\delta T}$ shares		
Long one forward		
Total		

Solution.

	Cash Flows	
	$t=0$	$t=T$
Selling $e^{-\delta T}$ shares	$S_0e^{-\delta T}$	$-P_T$
Long one forward	0	$P_T - F_{0,T}$
Total	$S_0e^{-\delta T}$	$-F_{0,T}$

Thus going short on a tailed position and buying a forward create cash flows like those of a risk-free short zero-coupon bond (i.e. borrowing) ■

If i is the rate of return on the synthetic bond then i satisfies the equation

$$S_0e^{(i-\delta)T} = F_{0,T}.$$

Solving for i we find

$$i = \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0e^{-\delta T}}\right) = \delta + \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right).$$

This rate is called the **implied repo rate**.

Remark 71.2
"Repo" is short for "repurchase." In the synthetic short bond, if we sell the stock short and enter into a long forward contract, then we have sold the stock and agreed to repurchase it. We have in effect taken a loan (shorted a bond), which must be repaid with interest at the implied repo rate.

Example 71.3
A $124 index pays a 1.5% continuous dividend. Suppose you observe a 2-year forward price of $135.7. Calculate the annual continuous rate of interest which you earn by buying stock and entering into a short forward contract, both positions for the same nominal amount.

Solution.
The implies repo rate is

$$i = \delta + \frac{1}{T}\ln\left(\frac{F_{0,T}}{S_0}\right) = 0.015 + \frac{1}{2}\ln\left(\frac{135.7}{124}\right) = 6\% \blacksquare$$

We next discuss how a market-maker uses the synthetic positions strategies to hedge his clients positions. Suppose that a client enters into a long forward position with forward price $F_{0,T}$ and expiration time T. In this case, the conterparty represented by the market-maker is holding a short forward position. To offset this risk, the market-maker borrows $S_0e^{-\delta T}$ and use this amount to buy a tailed position in the index, paying $S_0e^{-\delta T}$ and thus creating a synthetic long forward position. These transactions are summarized in the table below.

Table 71.1

	Cash Flows	
	$t = 0$	$t = T$
Buy tailed position in stock, paying $-S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	P_T
Borrow $S_0e^{-\delta T}$	$S_0e^{-\delta T}$	$-S_0e^{(r-\delta)T}$
Short forward	0	$F_{0,T} - P_T$
Total	0	$F_{0,T} - S_0e^{(r-\delta)T}$

A transaction in which you buy the asset and short the forward contract is called **cash-and-carry** (or **cash-and-carry hedge**). It is called cash-and-carry, because the cash is used to buy the asset and the asset is kept. A cash-and-carry has no risk: You have obligation to deliver the asset, but you also own the asset. An arbitrage that involves buying the asset and selling it forward is called **cash-and-carry arbitrage**.
Now suppose that a client wishes to enter into a short forward contract with forward price $F_{0,T}$ and expiration time T. In this case, the market-maker is holding a long forward position. To offset this risk, the market-maker short a tailed position in the index receiving $S_0e^{-\delta T}$ a tailed position in the index, paying $S_0e^{-\delta T}$ and thus creating a synthetic long forward position.

Example 71.4
A transaction in which you short the asset and long the forward contract is called **reverse cash-and-carry** (or **cash-and-carry hedge**). Complete the following table:

	Cash Flows	
	$t = 0$	$t = T$
Short tailed position in stock, receiving $S_0e^{-\delta T}$		
Lent $S_0e^{-\delta T}$		
Long forward		
Total		

Solution.
We have

Table 71.2

	Cash Flows	
	$t = 0$	$t = T$
Short tailed position in stock, receiving $S_0e^{-\delta T}$	$S_0e^{-\delta T}$	$-P_T$
Lent $S_0e^{-\delta T}$	$-S_0e^{-\delta T}$	$S_0e^{(r-\delta)T}$
Long forward	0	$P_T - F_{0,T}$
Total	0	$S_0e^{(r-\delta)T} - F_{0,T}$

Note that with a reverse cash-and-carry the market-maker is long a forward and short a synthetic forward contract ■

Note that if the forward contract is priced as $F_{0,T} = S_0e^{(r-\delta)T}$, then profits on a cash-and-carry or reverse cash-and-carry are zero. If $F_{0,T} \neq S_0e^{(r-\delta)T}$, an arbitrageur can do arbitrage. For example, if the forward contract is overpriced, that is $F_{0,T} > S_0e^{(r-\delta)T}$, than an arbitrageur can use the strategy described in Table 71.1 to make a risk-free profit. In contrast, if the forward contract is underpriced, that is $F_{0,T} < S_0e^{(r-\delta)T}$, than an arbitrageur can use the strategy described in Table 71.2 to make a risk-free profit.

Example 71.5
Suppose you are a market-maker in index forward contracts. The current price of an index is \$42.50. The index pays no dividends. You observe a 6-month long forward contract with forward price of \$43. The risk-free annual continuously compounded interest rate is 5%. Describe a strategy for creating an arbitrage profit and determine the amount of the profit.

Solution.
The no-arbitrage forward price of the long forward contract is $42.50e^{0.05(0.5)} = \$43.58$ which is larger than the observed long forward contract. We use the strategy of reverse cash-and-carry: We sell the index for \$42.50 and deposit the money into a savings account for six months. We also buy the forward contract. At the expiration date, the money in the bank grew to $42.50e^{0.05(0.5)} = 43.58$ and the cost of the forward is \$43. Thus, we make a profit of $43.58 - 43 = \$0.58$ ■

According to either Table 71.1 or Table 71.2 we have seen that an arbitrageur makes a costless profit only when $F_{0,T} \neq S_0e^{(r-\delta)T}$. This says that there is only one single no arbitrage forward price. In practice that is not the case. That is, there is a range of prices that preclude arbitrage opportunities. In fact, there is a no-arbitrage lower bound and upper bound between which there arbitrage can not be practiced. Transactions costs that determine these bounds are:
- Index ask and bid prices denoted by S_0^a and S_0^b with $S_0^b < S_0^a$.
- Forward contract ask and bid prices denoted by $F_{0,T}^a$ and $F_{0,T}^b$ with $F_{0,T}^b < F_{0,T}^a$.
- Index and forward transaction fees (commissions) denoted by C_S and C_F respectively.
- The difference between borrowing and lending rates $r^b > r^l$.

Let F^+ and F^- be the upper and lower bound respecively. Suppose that an arbitrageur believe that the forward price $F_{0,T}$ is too high. Assume that the index pays no dividends and that the arbitrageur applies the cash-and-carry strategy as illustrated in the table below.

	Cash Flows	
	$t = 0$	$t = T$
Buy tailed position in stock	$-S_0^a e^{-\delta T} - C_I$	P_T
Borrow $S_0^a e^{-\delta T} + C_I + C_F$	$[S_0^a e^{-\delta T} + C_I + C_F]$	$-[S_0^a e^{-\delta T} + C_I + C_F]e^{r^b T}$
Short forward	$-C_F$	$F_{0,T}^b - P_T$
Total	0	$F_{0,T}^b - [S_0^a e^{-\delta T} + C_I + C_F]e^{r^b T}$

An arbitrageur makes costless profit if $F_{0,T}^b > F^+ = [S_0^a e^{-\delta T} + C_I + C_F]e^{r^b T}$. This upper bound reflects the fact that we pay a high price for the index (the ask price), pay transaction costs on both the index and the forward, and borrow at a high rate. In our discussion, we assume no transaction fees at time T.
In Problem 71.14 we find the lower bound to be

$$F^- = (S_0^b - C_I - C_F)e^{r^l T}.$$

Example 71.6
Consider an index with current price of \$800 and with zero dividends. Suppose that an arbitrageur can borrow at the continuously compounded rate of 5.5% and lend at the continuously compounded rate of 5%. Suppose there is a \$1 transaction fee, paid at time 0, only for going either long or short a forward. Find the lower and upper bound between which the arbitrage is not profitable.

Solution.
We have $F^- = (800 - 1)e^{0.05} = \$839.97$ and $F^+ = (800 + 1)e^{0.055} = \$846.29$

Expected Future Price
Consider a forward contract with underlying asset a non-dividend paying stock. Suppose that the current price is S_0, the risk-free rate is r, and the delivery time is T. A forward price in this case is found to be $S_0 e^{rT}$. Does this mean that the stock price at time T is $S_0 e^{rT}$? The answer is no. That is, the forward price does not conveys any information about the future price of the stock. However, there is (conceptually) an expected rate of accumulation for the stock, call it α, that can be used to project an expected price for the stock at time $T : S_0 e^{\alpha T}$.
Since

$$\alpha = r + (\alpha - r)$$

the expected rate α has two components: the a risk-free component, r, representing compensation for the passage of time, and a risk component, $\alpha - r$, which represents compensation for assuming the risk of loss on the stock (risk premium). If $\alpha - r > 0$, then you must expect a positive return from the forward contract. The only way for this to happen is if the forward price predicts too low a stock price. In other words, the forward price a biased predictor of the future stock price.

Example 71.7 ‡
The current price of one share of XYZ stock is 100. The forward price for delivery of one share of XYZ stock in one year is 105. Which of the following statements about the expected price of one share of XYZ stock in one year is TRUE?
A. It will be less than 100
B. It will be equal to 100
C. It will be strictly between 100 and 105
D. It will be equal to 105
E. It will be greater than 105.

Solution.
In general, an investor should be compensated for time and risk. A forward contract has no investment, so the extra 5 represents the risk premium. Those who buy the stock expect to earn both the risk premium and the time value of their purchase and thus the expected stock value is greater than 100 + 5 = 105 and the answer is (E) ■

Cost of Carry and Lease Rate
Suppose you borrow S_0 at the continuously compounded risk-free rate r to purchase a share of stock that pays continuous dividends at the rate δ. At the end of one year, you will repay S_0e^r. However, the dividends will provide an offsetting income of S_0e^δ. Hence, at the end of one year you will have to pay $S_0e^{r-\delta}$. This is the net cost of carrying a long position in the stock. We call $r-\delta$ the **cost of carry**.
Now suppose you short the index and depositing the proceeds at a savings account paying continuously compounded annual rate r. Then at the end of one year you will earn S_0e^r. However, you would have to pay S_0e^δ to the index lender. Note that the index dividends are what the lender gives up by allowing the stock to be borrowed. We call δ the **lease rate** of the index: It is the annualized cash payment that the borrower must make to the lender.
With the definitions of the cost of carry and the lease rate, the forward pricing formula

$$F_{0,T} = S_0e^{(r-\delta)T}$$

has the following word interpretation

Forward price = Spot price + Interest to carry the asset - Lease rate

or

Forward price = Spot price + Cost of carry.

Example 71.8
What is the lease rate for a non-dividend paying stock? What is the cost of carry of a forward contract?

Solution.
For a non-dividend paying stock the lease rate is $\delta = 0$. A forward contract requires no investment and makes no payouts and therefore has a zero cost of carry ■

Practice Problems

Problem 71.1
A stock has a spot price of \$35. The continuously compounded risk free rate of interest is 6%. Find the prepaid forward price, the forward price and the annualized forward premium for a one-year forward contract on one share of the stock in each of the following situations.
(a) The stock pays no dividends.
(b) The stock pays a dividend of 1 in 6 months and 0.5 in one year just before delivery.
(c) The stock pays continuous dividends at a rate of 3.5%.

Problem 71.2
Suppose the stock price is \$55 and the 4-month forward price is \$57.50. Assuming the stock pays continuous dividends, find
(a) the 8-month forward premium
(b) the 8-month forward price.

Problem 71.3
Suppose the stock price is \$55 and the 4-month forward price is \$57.50. Assuming the stock pays continuous dividends, find
(a) Calculate the annualized forward premium.
(b) Calculate the 1-year forward price.

Problem 71.4
Consider the position of borrowing $S_0e^{-\delta T}$ at time 0 and selling a forward with forward price $F_{0,T} = S_0e^{(r-\delta)T}$. Complete the following table

	Cash Flows	
	$t = 0$	$t = T$
Short one forward		
Borrow $S_0e^{-\delta T}$		
Total		

Problem 71.5
Consider the position of selling $e^{-\delta T}$ shares of the index for $S_0e^{-\delta T}$ at time 0 and lending this amount for a period of T with interest rate r. Complete the following table

	Cash Flows	
	$t = 0$	$t = T$
Short tailed position in Stock		
Lend $S_0e^{-\delta T}$		
Total		

Problem 71.6
XYZ stock costs \$123.118 per share. This stock pays dividends at an annual continuous rate of 2.5%. A 18 month forward has a price of \$130.242. You own 10000 shares of XYZ stock. Calculate the annual continuous rate of interest at which you can borrow by shorting your stock.

Problem 71.7
You are a market-maker in S&R index forward contracts. The current price of the index is \$1100. The index pays no dividends. The risk-free annual continuously compounded interest rate is 5%.
(a) What is the no-arbitrage forward price of a 9-month forward contract.
(b) Suppose a customer wishes to enter into a long index forward position. If you take the opposite position, describe the strategy you would use to hedge the resulting short position.
(c) Suppose a customer wishes to enter into a short index forward position. If you take the opposite position, describe the strategy you would use to hedge the resulting long position.

Problem 71.8
Repeat the previous problem that index pays an 1.5% continuous dividend.

Problem 71.9
Suppose you are a market-maker in index forward contracts. The current price of an index is \$1100. The index pays no dividends. The risk-free annual continuously compounded interest rate is 5%.
(a) You observe a 6-month forward contract with forward price of \$1135. Describe a strategy for creating an arbitrage profit and determine the amount of the profit.
(b) You observe a 6-month forward contract with forward price of \$1115. Describe a strategy for creating an arbitrage profit and determine the amount of the profit.

Problem 71.10
Suppose you are a market-maker in index forward contracts. The current price of an index is \$1100. The index pays continuous dividends at 2%. The risk-free annual continuously compounded interest rate is 5%.
(a) You observe a 6-month forward contract with forward price of \$1120. Describe a strategy for creating an arbitrage profit and determine the amount of the profit.
(b) You observe a 6-month forward contract with forward price of \$1110. Describe a strategy for creating an arbitrage profit and determine the amount of the profit.

Problem 71.11
An ABC index has a current price of \$124 and the continuously compounded risk-free rate is 6%. Suppose you observe a 2-year forward price of \$135.7. What dividend yield is implied by this forward price?

Problem 71.12
Using the previous problem
(a) Suppose you believe that the dividend yield over the next two years will be 0.5%. What arbitrage would you take?
(b) Suppose you believe that the dividend yield over the next two years will be 3%. What arbitrage would you take?

Problem 71.13
Suppose the current stock price is \$45.34 and the continuously compounded intrest rate is 5%. The stock pays dividend of \$1.20 in three months. You observe a 9-month forward contract with forward price \$47.56. Is there an arbitrage opportunity on the forward contract? If so, describe the strategy to realize profit and find the arbitrage profit.

Problem 71.14
Suppose that an arbitrageur believes that the observed forward price $F_{0,T}$ is too low. Show that, for the arbitrageur to make costless profit we must have $F_{0,T}a < F^- = [S_0^a e^{-\delta T} - C_I - C_F]e^{r^l T}$.

Problem 71.15
Suppose that an arbitrageur would like to enter a cash-and-carry for 10000 barrels of oil for delivery in six months. Suppose that he can borrow at an annual effective rate of interest of 4.5%. The current price of a barrel of oil is \$55.
(a) What is the minimum forward price at which he would make a profit?
(b) What is his profit if the forward price is \$57?

Problem 71.16
Consider an index with current price of \$800 and with zero dividends. Suppose that an arbitrageur can borrow at the continuously compounded rate of 5.5% and lend at the continuously compounded rate of 5%. Suppose that there are no transaction fees. Then an arbitrage is not profitable if the forward price is between F^- and F^+. Determine the values of F^- and F^+.

Problem 71.17
Consider an index with current price of \$800 and with zero dividends. Suppose that an arbitrageur can borrow at the continuously compounded rate of 5.5% and lend at the continuously compounded rate of 5%. Suppose the transaction fee for going long or short a forward is \$1 and \$2.40 for going long or short an index. Find the lower and upper no-arbitrage bounds.

72 Futures Contracts

A **futures contract** is an agreement between two counterparts to buy/sell an asset for a specified price (the **future price**) at a specified date (the **delivery date**). Like the case of a forward contract, the holder of the contract is called the **long future** and the seller is called the **short future**. Futures are *exchange-traded* forward contracts. That is, they are bought and sold in organized future exhanges. Examples of exchanges the Chicago Mercantile Exchange, the Chicago Board of Trade, the International Petroleum Exchange of London, the New York Mercantile Exchange (NYMEX), the London Metal Exchange and the Tokyo Commodity Exchange. In the US, futures transactions are regulated by a government agency, the Commodity Futures Trading Commission (CFTC).

Each exchange has a **clearinghouse.** The role of the clearinghouse is to match the purchases and the sales which take place during the trading day. By matching trades, the clearinghouse never takes market risk because it always has offsetting positions with different counterparts. By having the clearinghouse as counterpart, an individual entering a futures contract does not face the possible credit risk of its counterpart.

Forwards and futures are similar in many aspects. However, there are still some differences which we list next.

• Forward contracts are privately negotiated and not standardized. They are traded mainly over the counter and can be customized to suit the buyer or the seller. In contrast, futures contracts are standardized and have specified delivery dates, locations, and procedures.

• Forward contracts are settled at the delivery date. Futures contracts are settled daily. As a result of daily settlements, value of a future contract is calculated according with the current value of the the underlying asset. We refer to this process as **marking-to-market.** Frequent marking-to-market can lead to pricing differences between futures contracts and identical forward contracts.

• Futures contracts are liquid. Unlike a forward contract, which requires settling with the counterparty by actually buying or selling the asset on the expiration date, a futures contract can be cancelled by entering into the opposite position. For example, if you are short a futures contract, you can cancel your obligation to sell by entering an offsetting obligation to buy the contract. This is done through a brokerage firm.

• In the presence of clearinghouse, futures contracts are structured so as to minimize the effects of credit risks. In contrast, in over-the-counter forward contracts, each party bears the other's credit risk.

• Because futures contracts are traded on an exchange, trades are subject to the exchange's rules, which typically include a limit on the amount of daily price changes by imposing temporary halts in trading if the price change exceeds a defined amount.

• Futures contracts are commonly cash-settled, rather than requiring delivery of the underlying asset.

We next illustrate an example of futures contracts with the S&P 500 index futures contract. The underlying asset for such a contract is the S&P 500 stock index. The contract has the following specifications:

• The futures contract is traded at the Chicago Mercantile Exchange.
• Notional Value: One contract has a value that is defined as the level of the index (let's say it is currently 1300) times \$250 (so an S&P 500 futures contract currently has a notional value of \$325,000).
• The contract has delivery months March, June, September and December.
• Trading ends on each business day prior to determination of settlement price. The settlement price is the average of the prices at which the contract traded immediately before the bell signaling the end of trading for the day.
• The contracts are cash-settled because it is inconvenient or impossible to deliver the underlying asset which in this case consists of a basket of 500 stocks. The cash settlement is based on the opening price of the index on the third Friday of delivery month.

Example 72.1
Suppose you wish to acquire \$2.2 million worth of S&P 500 index with futures price 1100.
(a) What is the notional value of one contract and how many futures contracts can you long for the invested amount?
(b) The **open interest** of a futures contract is defined to be the number of all futures outstanding contracts. It is the number of long positions. The activities of five traders are listed below. Complete the following table:

Date	Trading Activity	Open interest
Jan 1	A buys 1 contract and B sells 1 contract	
Jan 2	C buys 5 contracts and D sells 5 contracts	
Jan 3	A sells his contract and D buys one contract	
Jan 4	E buys 5 contracts from C who sells his 5 contracts	

Solution.
(a) The notional value of one futures contract is $250 \times 1100 = \$275,000$. With \$2.2 million you can long $\frac{2200000}{275000} = 8$ futures contract.
(b)

Date	Trading Activity	Open interest
Jan 1	A buys 1 contract and B sells 1 contract	1
Jan 2	C buys 5 contracts and D sells 5 contracts	6
Jan 3	A sells his contract and D buys one contract	6
Jan 4	E buys 5 contracts from C who sells his 5 contracts	6 ■

Margins and Marking to Market
One of the advantages of the futures contracts over the forward contracts is the dealing with credit risk. For example, forward contracts can be traded between two parties directly. This creates a credit risk related with the solvency of each party: One of the investors may regret the deal and try to back out or simply might not have the financial resources to honor the agreement. In contrast, in the case of futures contracts the exchange tries to organize trading in such a way to avoid credit defaults. This is when margins come in.
To avoid credit risk, a brokerage firm requires investors entering a futures contract to make deposit into an account called the **margin account** (also known as perfomance bond) which earns interest. The amount to be deposited at the time the contract is entered into is called the **initial margin** and is determined by the exchange. It is usually a fraction of the market value of the futures' underlying asset. The margin account is mainly intended to protect the counterparty of the contract.
Margin accounts are adjusted on a daily basis to reflect the investor's gain or loss. This practice is referred to as **marking to market** the account. A trade is marked to market at the close of the day on which it takes place. It is then marked to market at the close of the trading on each subsequent day.
If there exists a loss, the investor's broker transfers that amount from the investor's margin account to the clearinghouse. If a profit, the clearinghouse transfers that amount to investor's broker who then deposits it into the investor's margin account.
We illustrate the role of margins in the following example.

Example 72.2
On July 5, 2007, John instructs his broker to buy two gold futures contracts (each contract with **nominal amount** of 100 ounces of gold) on the New York Commodity Exchange (COMEX) with futures' price of \$400 per ounce. The annual continuously compounded rate of return is 1.5%.
(a) How many ounces of gold has John contracted to buy at the price of \$400 per ounce?
(b) What is the notional value of the two contracts?
(c) Suppose that the exchange requires 5% margin with daily settlement. What is the initial margin?
(d) Suppose that on July 6,2007, the futures price of gold has dropped from \$400 to \$397. What is the balance in John's margin account after settlement?
(e) Suppose that on July 7,2007, the futures price of gold has risen to \$400. What is the balance in John's margin account after settlement?

Solution.
(a) Since each contract consists of 100 ounces of gold, John has contracted to buy 200 ounces at the price of \$400 per ounce.
(b) The notional value of the two contracts is $2 \times 100 \times 400 = \$80,000$.
(c) The initial margin is $5\% \times 80,000 = \$4,000$.
(d) The balance in John's margin account after settlement is

$$4000e^{\frac{0.015}{365}} + 200(397 - 400) = \$3400.16.$$

(e) The balance in John's margin account after settlement is

$$34,000.16e^{\frac{0.015}{365}} + 200(400 - 397) = \$4,000.30 \blacksquare$$

An investor is entitled any balance in the margin account in excess of the initial margin. However, if the balance in an investors margin account falls, the broker (or clearinghouse) has less protection against default. Investors are required to keep the margin account to a minimum level which is a percentage of the initial margin. This minimum level is called the **maintenance margin.** When the balance in the margin account drops below the maintenance level the investor receives a **margin call** and is required to make a deposit to return the balance to the initial margin level. If the investor does not meet the margin call, the position is closed out and the investor receives any remaining balance in the margin account.

Example 72.3
A farmer enters into a short futures contract to sell 100,000 pounds of wheat for \$1.4 per pound. There is 30% margin and weekly settlement. The maintenance margin is 20% and the annual effective rate of interest is 4.5%. What is the minimum next week price which would lead to a margin call?

Solution.
The initial margin is

$$(0.30)(100000)(1.4) = \$42000.$$

The maintenance margin is

$$(0.20)(100000)(1.4) = \$28000.$$

After settlement, next week's balance is

$$42000(1.045)^{\frac{1}{52}} + 100000(1.4 - P_{\frac{1}{52}}).$$

A margin call happens if

$$28000 > 42000(1.045)^{\frac{1}{52}} + 100000(1.4 - P_{\frac{1}{52}}).$$

or

$$P_{\frac{1}{52}} > 1.4 - \frac{28000 - 42000(1.045)^{\frac{1}{52}}}{100000} = 1.540355672 \blacksquare$$

Practice Problems

Problem 72.1
The open interest on silver futures at a particular time is the number of
(A) all silver futures outstanding contracts
(B) outstanding silver futures contracts for a particular delivery month
(C) silver futures contracts traded during the day
(D) silver futures contracts traded the previous day

Problem 72.2
The fact that the exchange is the counter-party to every futures contract issued is important because it eliminates (A) market risk
(B) credit risk.
(C) interest rate risk

Problem 72.3
Margin must be posted by
(A) buyers of futures contracts
(B) sellers of futures contracts
(C) both buyers and sellers of futures contracts
(D) speculators only

Problem 72.4
The daily settlement of obligations on futures positions is called
(A) a margin call
(B) marking to market
(C) a variation margin check
(D) None of the above

Problem 72.5
All of the following are false except
(A) A margin deposit can only be met by cash
(B) All futures contracts require the same margin deposit
(C) The maintenance margin is the amount of money you post with your broker when you buy or sell a futures contract
(D) The maintenance margin is the value of the margin account below which the holder of a futures contract receives a margin call.

Problem 72.6
You are currently long in a futures contract. You then instruct a broker to enter the short side of a futures contract to close your position. This is called
(A) a cross hedge
(B) a reversing trade
(C) a speculation
(D) marking to market

Problem 72.7
A silver futures contract requires the seller to deliver 5,000 Troy ounces of silver. An investor sells one July silver futures contract at a price of \$8 per ounce, posting a \$2,025 initial margin. Neglecting interest, if the required maintenance margin is \$1,500, the price per ounce at which the investor would first receive a maintenance margin call is closest to:
(A) \$5.92
(B) \$7.89
(C) \$8.11
(D) \$10.80

Problem 72.8
If the initial margin on a futures contract I have sold increases, it likely means:
(A) The market has gotten less liquid.
(B) I am not making money in my futures positions.
(C) Prices have gotten more volatile.
(D) The margins never change once you have a futures position.

Problem 72.9
Complete the following statement: At the end of each trading day, the margin account is adjusted to reflect the futures trader's gain or loss. This process is referred to as ______________ the account.

Problem 72.10
What is the difference between the initial margin and the maintenance margin?

Problem 72.11
Explain some differences between a futures contract and a forward contract.

Problem 72.12
Crude oil futures trade in units of 1,000 U.S. barrels (42,000 gallons). The current value is \$50/barrel. Find the notional value of this contract.

Problem 72.13
Assume today's settlement price on a CME EUR futures contract is \$1.3140/EUR. The nominal amount of one contract is 125,000. You have a short position in one contract. Your margin account currently has a balance of \$1,700. The next three days' settlement prices are \$1.3126, \$1.3133, and \$1.3049. Calculate the changes in the margin account from daily marking-to-market and the balance in the account after the third day.

Problem 72.14
Do Problem 72.13 again assuming you have a long position in the futures contract. Assume 40% maintenance margin. Would you receive a margin call?

Problem 72.15
Suppose the S&P 500 index futures price is currently 1200. You wish to purchase four futures contracts on margin.
(a) What is the notional value of your position?
(b) Assuming a 10% initial margin, what is the value of the initial margin?

Problem 72.16
Suppose the S&P 500 index futures price is currently 950 and the initial margin is 10%. You wish to enter into 10 S&P 500 futures contracts.
(a) What is the notional value of your position? What is the margin?
(b) Suppose you earn a continuously compound rate of 6% on your margin balance, your position is marked to market weekly, and the maintenance margin is 80% of the initial margin. What is the greatest S&P 500 index futures price 1 week from today at which you will receive a margin call?

73 Understanding the Economy of Swaps: A Simple Commodity Swap

A **swap** is an agreement to exchange cash flows at specified future times according to certain specified rules. For example, a forward contract is a single-payment swap. To see this, suppose you enter a forward contract to buy 50 ounces of gold for \$300 an ounce in one year. You can sell the gold as soon you receive it. The forward contract acts like a single-payment swap because at the end of the year you will pay $50 \times 300 = \$15000$ and receive $50 \times P_1$ where P_1 is the market price of one ounce of gold on that date.
While a forward contract leads to an exchange of cash on a single future dates, swaps typically lead to cash flow exchanges taking place on several future dates. In some sense, a swap is a series of forward contracts combined into one contract or put it differently it can be regarded as a portfolio of forward contracts.
There are various types of swaps, namely, interest rates swap, currency swaps, commodity swaps, and equity swaps. In this section we will try to understand the economy of swaps. Our discussion will be centered around commodity swaps.
A **commodity swap** is a swap in which one party is selling a commodity (or the cash equivalent) and the other is paying cash for the commodity (the buyer). The contract needs to specify the type and quality of the commodity, how to settle the contract, etc. Commodity swaps are predominantly associated with energy-related products such as oil.

Suppose that quantity Q_i of the commodity is to be delivered at time t_i where $i = 1, 2, \cdots, n$. The buyer can either make one payment of C_0 at time 0 for the swap (a prepaid swap) or make a payment of C_i at time t_i. Let F_{0,t_i} be the forward price of the commodity with delivery at time t_i where $1 \le i \le n$. Let r_i be the annual interest rate applied over the interval of time $[0, t_i]$. Then the present value of the commodity delivered over the life of the swap is

$$\sum_{i=1}^{n} Q_i PV_i(F_{0,t_i}).$$

In a **prepaid swap**, the buyer makes a single payment today in the amount of

$$\sum_{i=1}^{n} Q_i PV_i(F_{0,t_i})$$

to obtain multiple deliveries in the future.

Example 73.1
A manufacturer is going to buy 100,000 barrels of oil 1 year from today and 2 years from today.

Suppose that the forward price for delivery in 1 year is \$20/barrel and in 2 years \$21/barrel. The 1-year spot interest rate is 6% and the 2-year spot interest rate is 6.5%.
The manufacturer enters into long forward contracts for 100,000 barrels for each of the next 2 years, committing to pay \$20/barrel in 1 year and \$21/barrel in 2 years. Alternatively, the manufacturer can make a single payment P now and the supplier would commit to deliver 1 barrel in year 1 and 1 barrel in year 2. Find P.

Solution.
P is just the cost of a prepaid swap given by

$$\frac{20}{1.06}+\frac{21}{1.065^2}=\$37.383 \blacksquare$$

Now, with multiple deliveries and multiple payments, the buyer makes swap payments at the time the commodity is delivered. Usually each swap payment per unit of commodity is a fixed amount say R. That is, at time t_i the swap payment is Q_iR. Thus, the present value of all the payments is given by the sum

$$\sum_{i=1}^{n} Q_iPV_i(R).$$

We thus obtain the equation

$$\sum_{i=1}^{n} Q_iPV_i(F_{0,t_i})=\sum_{i=1}^{n} Q_iPV_i(R)$$

From this equation, one finds R given by

$$R=\frac{\sum_{i=1}^{n} Q_iPV_i(F_{0,t_i})}{\sum_{i=1}^{n} Q_iPV_i(1)}.$$

Suppose that at time of delivery, the buyer pays a level payment of R at each of the delivery times. Then, the present value of the cashflow of payments is

$$\sum_{i=1}^{n} PV_i(R)$$

and R is given by

$$R=\frac{\sum_{i=1}^{n} Q_iPV_i(F_{0,t_i})}{\sum_{i=1}^{n} PV_i(1)}.$$

Example 73.2
Consider the information of Example 73.1. Suppose a swap calls for equal payments each year. Let x be the payment per year per barrel. This is the swap price. Find x.

Solution.
We must have
$$\frac{x}{1.06}+\frac{x}{1.065^2}=37.383.$$
Solving this equation for x we find $x=\$20.483$. This means, that each year, the manufacturer will pay \$20.483 and receives a barrel of oil ■

Remark 73.1
The payments need not be equal. Any payments that have a present value of \$37.383 are acceptable. That is, any two payments that satisfy the equation
$$\frac{P_1}{1.06}+\frac{P_2}{1.065^2}=\frac{20}{1.06}+\frac{21}{1.065^2}.$$

A swap can be settled either by physical settlement or by cash settlement. If a swap is settled physically, the party with the commodity delivers the stipulated notional amount to the buyer in exchange of cash as shown in the previous example.
If a swap is cashed settled, then the commodity is valued at the current spot price. If the current value of the commodity is larger than the value of the cash payment, then the buyer pays the swap counterparty the difference and then buys the commodity at the spot price. Reciprocally, if the current value of the commodity is smaller than the value of the cash payment, the counterparty pays this difference to the buyer and the buyer buys the commodity.

Example 73.3
Again, we consider the information of both previous examples. Suppose the swap is cash settled.
(a) Suppose that the market price is \$25. Calculate the payment which the buyer receives.
(b) Suppose that the market price is \$18. Calculate the payment which the counterparty receives.

Solution.
(a) The swap counterparty pays the buyer $25-20.483=\$4.517$ per barrel.
(b) The buyer pays the swap counterparty $18-20.483=-\$2.483$ per barrel ■

Notice that whatever the market price of oil, the net cost to the buyer is the swap price \$20.483:
$$\text{spot price} - (\text{spot price} - \text{swap price}) = \text{swap price}.$$

This formula describes the payments per one barrel of oil. For a swap agreement involving 100,000 barrels of oil, the values in the formula are multiplied by the **notional amount** of the swap (i.e. 100,000 barrels).

Remark 73.2
Notice that the results for the buyer are the same whether the swap is settled physically or financially. In both cases, the net cost to the buyer is \$20.483 per barrel.

A swap can be regarded as a series of forward contracts, combined with borrowing and/or lending money. In the above oil swap example, the buyer under the swap agreement would pay (and the counterparty would receive) the swap price of \$20.483 on each of the two dates. Under a pair of forward contracts, the forward prices of \$20 and \$21 would have been paid on the first and second dates, respectively. Relative to the forward prices, the buyer *lends* to the counterparty an amount of \$0.483 (= 20.483 − 20) on the first date by paying that much more than the forward rate; then the counterparty repays the loan by accepting \$20.483 on the second date, which is less than the forward price by \$0.517 (= 21 − 20.483). This loan has the effect of equalizing the net cash flow on the two dates.
The interest rate on the loan is

$$\frac{0.517}{0.483} - 1 = 7\%$$

This is the 1-year implied forward rate from year 1 to year 2.
A commodity swap allows to lock the price of a sale. It can be used by a producer of a commodity to hedge by fixing the price that he will get in the future for this commodity. It also can be used by a manufacturer to hedge by fixing the price that he will pay in the future for a commodity.

Swap Dealer
A dealer is an agent who makes a market buying and selling swaps to parties who have natural exposure to price risk. He earns his keep on the bid/ask spread.
The dealer buys a swap from the producer, at fixed price S (the swap price). The seller receives $S - P_T$, where P_T is the spot price. The dealer sells a swap to the consumer, at fixed price $S + \Delta$, where δ is the ask-bid spread. The buyer pays $S + \Delta - P_T$. The dealer earns difference between what buyer gives and seller gets, i.e. Δ.
The process of matching a buyer to the seller is called a **back-to-back** transaction. As you can see from the above discussion, the dealer is subject to each party's credit risk, but is not exposed to price risk.
Figure 73.1 illustrates the role of the swap dealer.

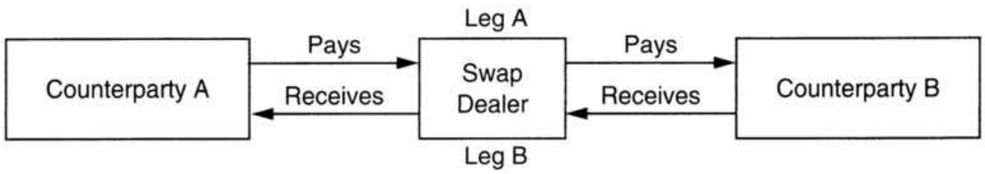


Figure 73.1

We next consider the case when the dealer serves as the counterparty. A dealer enters the swap as the counterparty. Thus, the dealer has the obligation to pay the spot price and receives the swap price. The dealer has a short position in 1- and 2- year oil. The dealer can lose money if the spot price increases. The dealer hedges his short position by going long year 1 and year 2 gold forwards. The table below illustrates how this strategy works.

Year	Payment from Gold Buyer	Long Forward	Net
1	$\$20.483 - P_1$	$P_1 - \$20$	$0.483
2	$\$20.483 - P_2$	$P_2 - \$21$	$-\$0.517$

The net cash flow for the hedged position is a loan where the dealer receives $0.483 in year 1 and repays $0.517 in year two. The dealer has interest rate exposure. In interest rate fall, the dealer will not be able to earn sufficient return from investing $0.483 in year 1 to repay $0.517 in year 2. This shows that hedging oil price risk does not fully hedge the position.

In summary, the dealer can eliminate its price risk by hedging the swap contract through the use of multiple forward contracts. The dealer must also enter into forward rate agreements in order to hedge the interest rate exposure resulting from the timing difference between the forward contract's payments and the payments under the swap agreement.

The Market Value of a Swap

When a swap agreement is created, its market value is zero. In our oil example, the swap consists of two forward contracts and an agreement to lend money at the implied forward rate of 7%. These forwards have zero values at time 0 so the swap does as well. As time passes the market value of the swap will no longer be zero due to oil prices and interest rates change. So over time, the swap contract develops a non-zero value. Even if prices and interest rates do not change, the swap agreement will have a non-zero value once the first payment has been made.

Let us find the market value of the oil swap example. Suppose that right after entering the swap the oil prices go up by $2 in year 1 and year 2. The original swap in this case has a non-zero market value. Assuming interest rate are the same, the new swap price x satisfies the equation

$$\frac{x}{1.06} + \frac{x}{1.065^2} = \frac{22}{1.06} + \frac{23}{1.065^2}$$

which gives $x = \$22.483$. In this case, the buyer buys a barrel of oil for \$20.483 and can sell it for \$22.483 making a net profit of \$2 per barrel each year. The present value of this difference of prices is

$$\frac{2}{1.06} + \frac{2}{1.065^2} = \$3.65.$$

The buyer can receive a stream of payments worth \$3.65 by offsetting a new swap. Thus, \$3.65 is the market value of the swap.
If interest rates had changes, we would have used the new interest rate in computing the new swap price.

The swap market is very large and has no major governmental oversight. Swap markets have none of the provisions as the futures and options markets do, thus resulting in a default risk and the need to assess the creditworthiness of each counterparty. Traders must carefully assess their trade partners.
On the other side of this, swaps have a measure of privacy and can be negotiated for unique quantities, qualities, and contract provisions.

Example 73.4
Suppose that oil forward prices for 1 year, 2 years, and 3 years are \$20, \$21, and \$22. The effective annual interest rates are shown in the table below

Maturity(in years)	yield %
1	6
2	6.5
3	7

(a) What is the swap price, assuming level payments?
(b) Suppose you are a dealer who is paying the swap price and receiving the spot price. Suppose you enter into the swap and immediately thereafter all interest rates rise by 0.5% with the oil forward prices remain unchanged. What happens to the value of your swap position?
(c) What if interest rates fall by 0.5%?
(d) What hedging instrument would have protected you against interest rate risk in this position?

Solution.
(a) The swap price x satisfies the equation

$$\frac{x}{1.06} + \frac{x}{1.065^2} + \frac{x}{1.07^3} = \frac{20}{1.06} + \frac{21}{1.065^2} + \frac{22}{1.07^3}.$$

Solving for x we find $x = \$20.952$.
(b) You should short 3 forward contracts. This strategy is illustrated in the table below.

Year	Payment from Gold Buyer	Short Forward	Net
1	$P_1 - \$20.952$	$\$20 - p_1$	$-\$0.952$
2	$P_2 - \$20.952$	$\$21 - p_2$	$\$0.048$
3	$P_3 - \$20.952$	$\$22 - p_2$	$\$1.048$

With the 0.5% increase in rates we find that the net present value of the loan is

$$\frac{-0.952}{1.065} + \frac{0.048}{1.07^2} + \frac{1.048}{1.075^3} = -\$0.0081.$$

The dealer never recovers from the increased interest rate he faces on the overpayment of the first swap payment.
(c) If interest rates fall by 0.5% then the net present value of the loan is

$$\frac{-0.952}{1.055} + \frac{0.048}{1.06^2} + \frac{1.048}{1.055^3} = \$0.0083.$$

The dealer makes money, because he gets a favorable interest rate on the loan he needs to take to finance the first overpayment. (d) The dealer could have tried to hedge his exposure with a forward rate agreement or any other derivative protecting against interest rate risk ■

Practice Problems

Problem 73.1
Match (a)-(d) with (I)-(IV)

(a) Notional amount is
(b) The contracting parties in a swap are called
(c) What the swap dealers receive for their services is
(d) The ask-spread commission is

(I) ask-spread bid
(II) counterparties
(III) the difference between the cash flow one counterparty pays to the dealer and what dealer pays out to the other counterparty
(IV) used to calculate the payments that will be exchanged between the two parties.

Problem 73.2 ‡
Zero-coupon risk-free bonds are available with the following maturities and yield rates (effective, annual):

Maturity(in years)	yield %
1	6
2	6.5
3	7

You need to buy corn for producing ethanol. You want to purchase 10,000 bushels one year from now, 15,000 bushels two years from now, and 20,000 bushels three years from now. The current forward prices, per bushel, are \$3.89, \$4.11, and \$4.16 for one, two, and three years respectively.
You want to enter into a commodity swap to lock in these prices. Which of the following sequences of payments at times one, two, and three will NOT be acceptable to you and to the corn supplier?
(A) \$38,900; \$61,650; \$83,200
(B) \$39,083; \$61,650; \$82,039
(C) \$40,777; \$61,166; \$81,554
(D) \$41,892; \$62,340; \$78,997
(E) \$60,184; \$60,184; \$60,184

Problem 73.3
A manufacturer uses oil in his profession. He needs 10,000 barrels in two months, 12,000 barrels in four months, and 15,000 barrels in six months. The manufacturer enters into a long oil swap. The

payment of the swap will be made at the delivery times.
The current forward prices, per barrel, are \$55, \$56, and \$58 for two, four, and six months respectively. Zero-coupon risk-free bonds are available with the following maturities and yield rates (nominal, convertible monthly):

Maturity(in months)	yield %
2	4.5
4	4.55
6	4.65

(a) Suppose level payments of R are made at each of the delivery times. Calculate R.
(b) Suppose that the cash swap payment consists of a unique payment per barrel made at each of the delivery times. Calculate the no arbitrage swap price per barrel.

Problem 73.4
Suppose that a transportation company must buy 1000 barrels of oil every six months, for 3 years, starting 6 months from now. Instead of buying six separate long forward contracts, the company enters into a long swap contract. The payment of the swap will be made at the delivery times.
The current forward prices, per barrel, are \$55, \$57, \$57, \$60, \$62, and \$64 for 6-, 12-, 18-, 24-, 30-, and 36- months respectively. Zero-coupon risk-free bonds are available with the following maturities and yield rates (nominal, convertible semiannually):

Maturity(in months)	yield %
6	5.5
12	5.6
18	5.65
24	5.7
30	5.7
36	5.75

(a) With level payments, find the price per barrel of oil using the swap.
(b) Suppose the swap is settled in cash. Assume that the spot rate for oil in 18 months is \$57. Calculate the payment which the counterparty receives.
(c) Suppose that immediately after the swap is signed up, the future prices of oil are given by the table

Forward Price	\$55	\$58	\$59	\$61	\$62	\$63
Expiration (in months)	6	12	18	24	30	36

Find the market value of the swap for the company.

Problem 73.5
Suppose that a jewelry manufacturer needs to purchase 100 ounces of gold one year from now and another 100 ounces in two years, and that the forward price for gold is \$750 per ounce for delivery in one year and \$800 for delivery in two years. The manufacturer could enter into two long forward contracts for these two dates at these two prices, and eliminate the risk of price variation. Alternatively, a single swap agreement could be used to fix a single price for the purchases on these two dates. Suppose that the 1-year and 2-year spot rates are 5% and 6% respectively.
(a) With level payments, find the price per ounce of gold using the swap.
(b) Suppose that the payments are not leveled, say with payment P_1 per ounce for year 1 and P_2 for year 2. Set the equation that P_1 and P_2 must satisfy.
(c) What is the notional amount of the swap?
(d) The buyer pays an amount larger than the forward price in 1 year. The difference is considered as a loan for the counterparty. What is the value of this amount per ounce?
(e) The buyer pays an amount smaller than the forward price in 2 years. The difference is considered as the counterparty repayment of the loan. What is the value of this amount per ounce?
(f) What is the implied forward rate from year 1 to year 2 of the loan?

Problem 73.6
Consider the information of the previous problem. Suppose the forward prices are \$780 and \$830. The 1- and 2-year interest rates are 5% and 6% respectively. Find the new swap price.

Problem 73.7
Suppose that gold forward prices for 1 year, 2 years, and 3 years are \$750, \$800, and \$850. The effective annual interest rates are shown in the table below

Maturity(in years)	yield %
1	6
2	6.5
3	7

(a) What is the swap price?
(b) What is the price of a 2-year swap with the first settlement two years from now and the second in 3 years?

Problem 73.8
Consider the 2-year gold swap of Problem 73.5. Suppose a dealer is paying the spot price and receiving the swap price. What position in gold forward contracts will hedge gold price risk in this position?

Problem 73.9
Consider the 2-year gold swap of Problem 73.5. Suppose a dealer is paying the swap price and receiving the spot price. What position in gold forward contracts will hedge gold price risk in this position?

74 Interest Rate Swaps

In this section we consider another type of swaps: the interest rate swaps. Companies use **interest rate swap** as a means of converting a series of future interest payments that vary with changes in interest rates (**floating** payments) to a series of **fixed** level payments (i.e. **swap rate**), or vice-versa. The fixed stream of payments are computed with respect to a rate determined by the contract. The floating stream of payments are determined using a benchmark, such as the LIBOR[1] The amount in which the interest payments is based is called the **notional principal**. The life of a swap is called the **swap term** or **swap tenor.**

Interest rate swaps are used to hedge against or speculate on changes in interest rates: They are often used by firms to alter their exposure to interest-rate fluctuations, by swapping fixed-rate obligations for floating rate obligations, or vice versa. By swapping interest rates, a firm is able to alter its interest rate exposures and bring them in line with management's appetite for interest rate risk. They are also used speculatively by hedge funds or other investors who expect a change in interest rates or the relationships between them. Interest rate swaps are also very popular due to the arbitrage opportunities they provide.

The interest rate swap market is closely linked to the Eurodollar[2] futures market which trades at the Chicago Mercantile Exchange.

All interest rate swaps payments are settled in net: If the fixed rate is greater than the floating rate, the buyer pays the counterparty the difference. In contrast, if the fixed rate is smaller than the floating rate, the counterparty pays the buyer the difference.

The interest rate determination date for the floating interest payment occurs at the beginning of the period with payment due at the end of the period. It is important to keep in mind that there is no exchange of the notional principal in a swap.

Example 74.1

On December 31, 2006, firm ABC enters into an interest rate swap with firm XYZ. The notional principal is \$100 m. Firm ABC pays the fixed rate of 5% compounded annually and receives floating rate from firm XYZ at a 12-month LIBOR. The swap term is 5 years. That is, the two parties exchange payments annually on December 31, beginning in 2007 and concluding in 2011. This swap is illustrated in Figure 74.1

[1] The (**London Interbank offer rate**) LIBOR is the most widely used reference rate for short term interest rates world-wide. The LIBOR is published daily the (British Bankers Association) BBA. It is based on rates that large international banks in London offer each other for interbank deposits. Rates are quoted for 1-month, 3-month, 6-month and 12-month deposits.

[2] Eurodollars are US dollars deposited at banks outside the United States, primarily in Europe. The interest rate paid on Eurodollars is largely determined by LIBOR. Eurodollar futures provide a way of betting on or hedging against future interest rate changes.

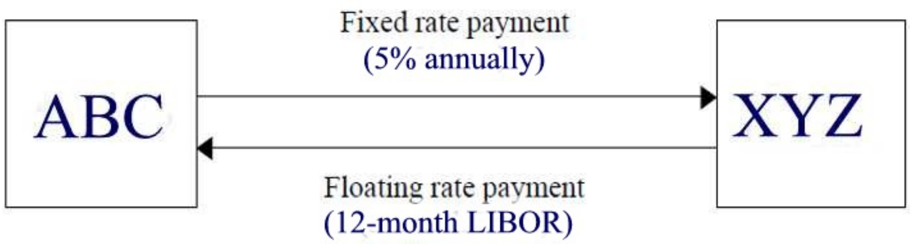


Figure 74.1

(a) On December 31,2006 the 12-month LIBOR rate was 4.5%. Determine the amount paid by ABC to XYZ on December 31,2007.
(b) On December 31,2006 the 12-month LIBOR rate was 5.5%. Determine the amount received by ABC from XYZ on December 31,2007.

Solution.
(a) On December 31,2007 ABC will pay XYZ the amount $\$100\text{m}\times 0.05 = \$5,000,000$ and receives from XYZ the amount $\$100\text{m}\times 0.045 = \$4,500,000$. Thus, on December 31,2007, ABC pays XYZ the net payment of $500,000.
(a) On December 31,2007 ABC will pay XYZ the amount $\$100\text{m}\times 0.05 = \$5,000,000$ and receives from XYZ the amount $\$100\text{m}\times 0.055 = \$5,500,000$. Thus, on December 31,2007, ABC receives from XYZ the net amount of $500,000.
On net, ABC is paying At no point does the principal change hands, which is why it is referred to as a *notional* amount ■

Remark 74.1
A fixed-for-floating interest rate swap is often referred to as a **plain vanilla** swap because it is the most commonly encountered structure.

Counterparty Risk of a Swap
Unlike future markets, swaps are over-the-counter instruments and so they are not backed by the guarantee of a clearing house or an exchange. If one party defaults, they owe to the other party at most the present value of net swap payments that they are obligated to make at current market prices.

Asset Swap
An **asset swap** is an interest rate swap used to convert the cash flows from and underlying security (such a bond) from fixed coupon to floating coupon. Thus, an asset swap converts a fixed-rate bond

to a synthetic floating-rate bond. The asset swap transaction is illustrated in Figure 74.2.

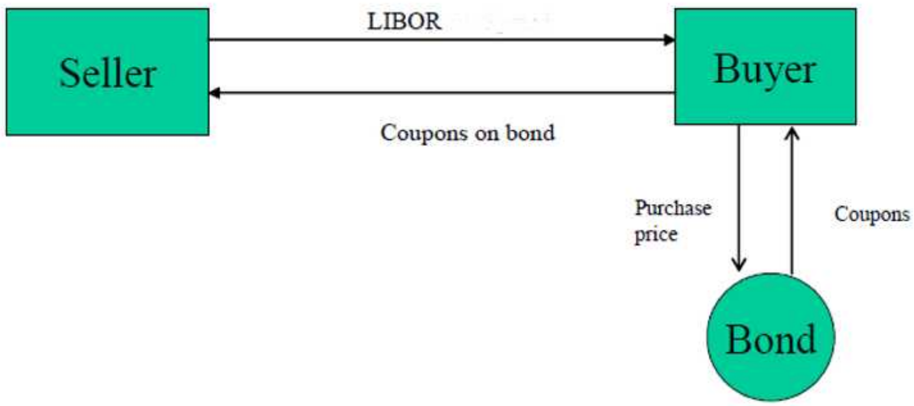


Figure 74.2

Asset swaps are commonly used by fund managers who own fixed-rate bonds and wish to have floating-rate exposure while continuing to own the bonds.

Computing the Swap Rate

Consider an interest rate swap with n settlements occurring on dates $t_i, i = 1, 2, \cdots, n$. Let $P(0, t_i)$ be the price of a \$1 face value zero-coupon bond maturing on date t_i. Notice that $P(0, t_i)$ is the discount factor from time zero to time t_i based on the t_i-year spot rate. That is, if i_t is the $t-$year spot rate then $P(0, t) = (1 + i_t)^{-t}$. Let $r_0(t_{i-1}, t_i)$ be the implied forward rate between the times t_{i-1} and t_i. From Section 50, we have

$$r_0(t_{i-1}, t_i) = \frac{P(0, t_{i-1})}{P(0, t_i)} - 1.$$

An interest rate swap acts much like the oil price swap described in the previous section. When initiated, the swap has a value of zero, i.e., the fixed rate payments and the variable rate payments have equal present values. In terms of a formula, we have:

$$R \sum_{i=1}^{n} P(0, t_i) = \sum_{i=1}^{n} P(0, t_i) r_0(t_{i-1}, t_i)$$

where R denote the fixed swap rate. Solving this last equation for R we obtain

$$R = \frac{\sum_{i=1}^{n} P(0, t_i) r_0(t_{i-1}, t_i)}{\sum_{i=1}^{n} P(0, t_i)}$$

where $\sum_{i=1}^n P(0,t_i)r_0(t_{i-1},t_i)$ is the present value of the variable payments and $\sum_{i=1}^n P(0,t_i)$ is the present value of a \$1 annuity when interest rates vary over time.
Note that the swap rate can be interpreted as a weighted average of the implied forward rates, where zero-coupon bond prices are used to determine the weights.
Now, since

$$r_0(t_{i-1},t_i) = \frac{P(0,t_{i-1})}{P(0,t_i)} - 1$$

we can write

$$\sum_{i=1}^n P(0,t_i)r_0(t_{i-1},t_i) = \sum_{i=1}^n [P(0,t_{i-1}) - P(0,t_i) = 1 - P(0,t_n)$$

so that

$$R = \frac{1-P(0,t_n)}{\sum_{i=1}^n P(0,t_i)}$$

This formula can be rearranged to obtain

$$P(0,t_n) + R\sum_{i=1}^n P(0,t_i) = 1.$$

This formula states that a payment of 1 that will be received n years from now, plus a payment of R at the end of each of the next n years, has a total present value of 1. This is the valuation equation for a bond priced at par (i.e. face value \$1) with a coupon rate R. Thus, the swap rate is the coupon rate for a bond that is selling at par.

Example 74.2
The following table lists prices of zerocoupon \$1face value bonds with their respective maturities:

Years to Maturity	Price
1	\$0.956938
2	\$0.907029
3	\$0.863838

(a) Calculate the 1-, 2-, and 3-year spot rates of interest.
(b) Calculate the 1- and 2-year forward rates of interest.
(c) Calculate the coupon rate R for a 3-year bond with annual coupons whose face value, redemption value and price are all one.

Solution.
(a) Let i_1 be the spot rate of interest. Then $1+i_1 = P(0,1)^{-1} = 0.956938^{-1} \to i_1 = 0.956938^{-1} -$

$1 = 4.499967135\%$. Likewise, $i_2 = 0.907029^{\frac{-1}{2}} - 1 = 5.000027694\%$ and $i_3 = 0.863838^{\frac{-1}{3}} - 1 = 4.99991613\%$
(b) The 1-year forward rate is given by

$$r_0(1,2) = \frac{P(0,1)}{P(0,2)} - 1 = \frac{0.956938}{0.907029} - 1 = 5.502470153\%.$$

Likewise, the 2-year forward rate is given by

$$r_0(2,3) = \frac{P(0,2)}{P(0,3)} - 1 = \frac{0.907029}{0.863838} - 1 = 4.999895814\%.$$

(c) The coupon rate is given by

$$R = \frac{1 - P(0,3)}{P(0,1) + P(0,2) + P(0,3)} = \frac{1 - 0.863838}{0.956938 + 0.907029 + 0.863838} = 4.991632466\% \blacksquare$$

Deferred Swaps
By a **k-deferred swap** we mean a swap whose exchange of payments start k periods from now. The (deferred) swap rate is applied in the future but it is agreed upon today. It is computed as follows

$$\begin{aligned} R =& \frac{\sum_{j=k}^{n} P(0,t_j) r_0(t_{j-1},t_j)}{\sum_{j=k}^{n} P(0,t_j)} \\ =& \frac{\sum_{j=k}^{n} P(0,t_j)\left(\frac{P(0,t_{j-1})}{P(0,t_j)} - 1\right)}{\sum_{j=k}^{n} P(0,t_j)} \\ =& \frac{\sum_{j=k}^{n}(P(0,t_{j-1}) - P(0,t_j)}{\sum_{j=k}^{n} P(0,t_j)} \\ =& \frac{P(0,t_{k-1}) - P(0,t_n)}{\sum_{j=k}^{n} P(0,t_j)} \end{aligned}$$

Example 74.3
Given the following zero-coupon bond prices:

Quarter	1	2	3	4	5	6	7	8
$P(0,t)$	0.9852	0.9701	0.9546	0.9388	0.9231	0.9075	0.8919	0.8763

What is the fixed-rate (deferred swap rate) in a 6-quarter interest rate swap with the first settlement in quarter 3?

Solution.
We have

$$\begin{aligned}
R =& \frac{P(0,2) - P(0,8)}{\sum_{j=3}^{8} P(0,j)} \\
=& \frac{0.9701 - 0.8763}{0.9546 + 0.9388 + 0.9231 + 0.9075 + 0.8919 + 0.8763} \\
=& 1.71\% \blacksquare
\end{aligned}$$

The Value of a Swap
At initiation of the swap, the value of the swap is set to be zero to both parties. The fixed rate of the swap is then calculated such that the present value of the fixed payments equals that of the floating payments. As time evolves, the interest rate may move upward or downward so that the value of the floating payments changes. Therefore, the value of the swap changes in later times, and its value at each time is given by the difference in the present values of the remaining cash flows from the two parties.
Even in the absence of changes in interest rates, as soon as the first swap payment occurs, the swap will have a value. The swap will have a positive value for the party who paid too much (e.g., the fixed rate payer if the fixed rate exceeds the variable rate on the first payment date), and a negative value for the party who received the net payment.
As with the commodity swap, an interest rate swap is equivalent to entering into a series of forward contracts and also undertaking some borrowing and lending.

Practice Problems

Problem 74.1
Interest rate swaps allow one party to exchange a:
(A) floating interest for a fixed rate over the contract term.
(B) fixed interest rate for a lower fixed rate over the contract term.
(C) floating interest rate for a lower floating rate over the contract term.

Problem 74.2
All of the following are types of financial instruments except for:
(A) currency forward contracts
(B) currency futures contracts
(C) currency options
(D) swap agreements
(E) money-market hedge

Problem 74.3
On March 5,2004, an agreement by Microsoft to receive 6-month LIBOR and pay a fixed rate of 5% per annum every 6 months for 3 years on a notional principal of $100 million. Complete the following table:

Date	6-month LIBOR rate	Fixed Payment	Floating Payment	Net Cash Flow
March 5,2004	4.2%			
Sept 5,2004	4.8%			
March 5, 2005	5.3%			
Sept 5, 2005	5.5%			
March 5, 2006	5.6%			
Sept 5, 2006	5.9%			
March 5,2007	6.4%			

Problem 74.4
Firm ABC is paying $750,000 in interest payments a year while Firm XYZ is paying LIBOR plus 75 basis points (i.e. 0.75%) on $10,000,000 loans. The current LIBOR rate is 6.5%. Firm ABC and XYZ have agreed to swap interest payments, how much will paid to which Firm this year?

Problem 74.5
Given the following information

t (in months)	$P(0,t)$
3	0.986923
6	0.973921
9	0.961067
12	0.948242

Calculate the annual nominal interest rate compounded quarterly for a loan with the following maturity dates: 3, 6, 9 and 12 months.

Problem 74.6
Given the following information

t (in years)	$P(0,t)$
1	0.95785
2	0.91049
3	0.85404

Calculate the 3-year interest rate swap.

Problem 74.7
The fixed rate for a 2-year interest rate swap is 6.06%. The 2-year spot rate is 6.10%. What is the price of a 1-year zero-coupon bond with a maturity value of 1,000?

Problem 74.8
Given the following zero-coupon bond prices:

Quarter	1	2	3	4	5	6	7	8
$P(0,t)$	0.9852	0.9701	0.9546	0.9388	0.9231	0.9075	0.8919	0.8763

What is the fixed-rate (deferred swap rate) in a 5-quarter interest rate swap with the first settlement in quarter 2?

Problem 74.9
Given the following zero-coupon bond prices:

Quarter	1	2	3	4	5	6	7	8
$P(0,t)$	0.9852	0.9701	0.9546	0.9388	0.9231	0.9075	0.8919	0.8763

What is the swap rate in an 8-quarter interest rate swap?

Problem 74.10
The following table lists prices of zerocoupon $1face value bonds with their respective maturities:

Years to Maturity	Price
1	\$0.95238
2	\$0.89845
3	\$0.842.00

Complete the following table:

Years to Maturity	Price	Spot Rate i_n	$r_0(n-1,n)$
1	\$0.95238		
2	\$0.89845		
3	\$0.842.00		

Problem 74.11

Using the data in the previous table, find the fixed rate (i.e. the swap rate R).

75 Risk Management

Risk management consists of identifying the sources of risk, choosing the ones to be hedged, and choosing the way of hedging. A firm's risk management consists of the use of financial derivatives such as forwards, calls, and puts to alter its exposure to risk and protect its profitability.
We will next discuss risk management from both the producer's perspective as well as the buyer's perspective.

The Producer's Perspective
A producer normally has a long position that consists of its product. In order to protect its future profits from price fluctuations, the producer can offset his long position by using forward contracts, puts options, call options, or some combination of these derivatives. This process is also referred to as **hedging** or **insuring** the long position.
Some of the risk management strategies for a producer include the following:

(1) **Hedging with a forward contract:** In order for the producer to lock in the price of his product, the producer can enter into a short forward contract that fixes the future sale price at the current forward price. There is no initial cost for the producer. For example, ABC Inc, an oil producing firm, enters into a short forward contract, agreeing to sell oil at today's current price of $65 a barrel in one year.

(2) **Hedging with a put option:** Creating a floor by purchasing a put option which limits the losses if the price declines, but allows unlimited profit if the price rises. Note that the option's premium is a cost to the producer for creating the floor. For example, ABC Inc purchases 65-strike put at a premium of $3.50 per barrel. In this case, the put option plays the role of an insurance that guarantees a minimum price of $65 for a barrel of oil.

(3) **Insuring by selling a call:** A written call (selling a cap) sets a maximum price (and therefore limit profit) for the product if the price rises, but does not limit losses if the price declines. The premium received by the producer helps reduce the losses. For example, ABC Inc sells a 65-strike call for a premium of $3.50 per barrel.

(4) **Creating a collar** by purchasing a put at one strike price and writing a call at a higher strike price. A collar sets maximum and minimum prices (the call strike and the put strike, respectively) that the producer may realize for its product. The producer is exposed to the risk of variation between these two prices, but is not affected by price variation above or below this range. The put and call options constitute the collar, but in combination with the producer's long position in the product, they form a bull spread.

Example 75.1
ABC is a coppermining company with fixed costs \$0.75/lb and variable costs of \$2.25/lb.
(a) If ABC does nothing to hedge price risk, what is its profit 1 year from now, per pound of copper?
(b) Suppose that ABC can enter a short forward contract agreeing to sell its copper production one year from now. The 1-year forward price is \$3.5/lb. What is the estimated profit one year from now?
(c) Suppose that ABC would like to benefit if the price of the copper goes higher than \$3.5/lb. It could use options. One year option prices for copper are given below

Strike	3.3	3.4	3.5	3.6	3.7	3.8	3.9
Call	0.42567	0.3762	0.33119	0.29048	0.25386	0.22111	0.19196
Put	0.23542	0.28107	0.33119	0.3856	0.44411	0.50648	0.57245

The 1-year continuously compounded interest rate is 5%. Compute the estimated profit in 1 year if ABC buys put options with a strike of \$3.7/lb and with a strike of \$3.4/lb.
(d) Compute the estimated profit in 1 year if ABC sells call options with a strike of \$3.7/lb and with a strike of \$3.4/lb.
(e) Compute the profit in 1 year if ABC buys a 3.6-strike put and sells a 3.9-strike call.

Solution.
(a) The profit is the one-year spot price minus total cost. That is, $P_1 - 3$. A graph of the unhedged position is shown in Figure 75.1.
(b) In this case, the profit is $P = P_1 - 3 + (3.5 - P_1) = \$0.5$. A graph of the hedged position is shown in Figure 75.1. For an uninsured position, the possible losses can be very high. However, by entering the forward, the company has a fixed benefit.

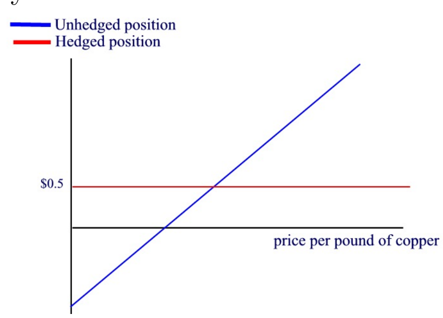


Figure 75.1

(c) The profit per pound that results from buying the 3.7-strike put is

$$P_1 - 3 + \max\{3.7 - P_1, 0\} - 0.44411e^{0.05}.$$

The graph of the profit is given in Figure 75.2.

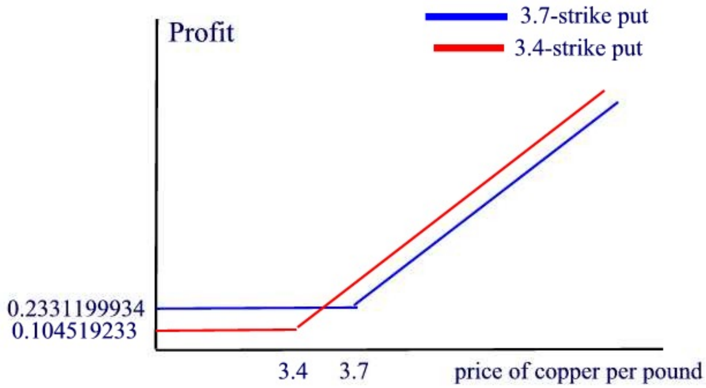


Figure 75.2

The profit per pound that results from buying the 3.4-strike put is

$$P_1 - 3 + \max\{3.4 - P_1, 0\} - 0.28107e^{0.05}.$$

The graph of the profit is given in Figure 75.2. With the 3.4-strike put, the company makes \$0.17139924/lb more than the 3.7-strike put if the price of copper is over \$3.7/lb. However, its guaranteed profit is \$0.104519233, which is smaller than the guaranteed profit under a 3.7-strike put. Buying a put is like buying insurance against small spot prices. The larger the strike price the larger the price of the insurance. A producer company needs to buy a put with a strike large enough to cover low spot prices. If the strike put is too large, the company will be wasting money in insurance which it does not need.
(d) The profit per pound that results from selling the 3.7-strike call is

$$P_1 - 3 - \max\{P_1 - 3.7, 0\} + 0.25386e^{0.05}.$$

The graph of the profit is given in Figure 75.3.

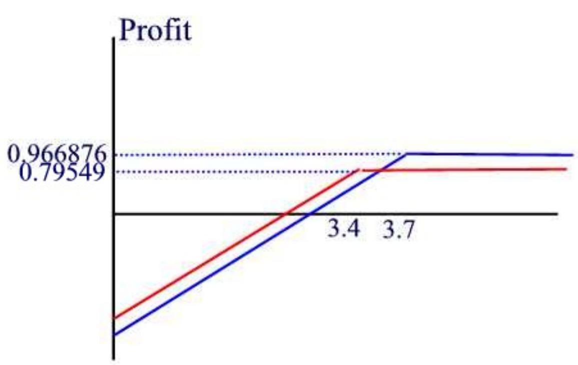


Figure 75.3

The profit per pound that results from selling the 3.4-strike call is

$$P_1 - 3 - \max\{P_1 - 3.4, 0\} + 0.3762e^{0.05}.$$

The graph of the profit is given in Figure 75.3.
(e) The profit is

$$S_1 - 3 + \max\{3.6 - S_1, 0\} - \max\{3.4 - S_1, 0\} - (0.3856 - 0.19196)e^{0.05}.$$

Under this strategy the profit of the company is very close to that of a forward contract. But, instead of winning a constant of \$0.5/lb. in the forward contract, the company's profit varies with the future price of copper, although not much ■

The Buyer's Perspective
A buyer of the producer's product is in the opposite position from the producer. The buyer can engage in any of the above risk management strategies, but would do the opposite of what the producer does. The buyer's possible strategies are therefore as follows:
- Enter into a long forward contract.
- Sell a put option (floor).
- Buy a call option (cap).

Example 75.2
MBF is a wire manufacturer. It buys copper and manufactures wires. Suppose the fixed cost of a unit of wire is \$2 and the noncopper variable cost is \$1.25. A unit of wire sells for \$5 plus the price of copper. One pound of copper is needed to manufacture one unit of wire.
(a) What is the unhedged profit of MBF?
(b) Suppose that MBF buys a forward contract with forward price of \$3.5/lb. What is the profit one year from now?

Solution.
(a) MBF needs copper to manufacture its product. Thus, an increase in copper price results in an increase in cost of making the wire. So one may think that the company needs to hedge for the price fluctuations. But since the company will adjust its selling price to reflect the change in the copper price, this chnage will cancel the copper price risk and as a result there is no need for hedging.
The unhedged profit per pound of copper in one year from today is

$$\text{Revenue} - \text{Cost} = P_1 + 5 - (2 + 1.25 + P_1) = 1.75.$$

We see that the profits of MBF do not depend on the price of copper. Cost and revenue copper price risk cancel each other out.
(b) In this case, the profit per pound of copper one year from today is given by

$$P_1 + 5 - (2 + 1.25 + P_1) + P_1 - 3.5 = P_1 - 1.75.$$

Note that buying a forward introduces price risk ■

Example 75.3
Alumco is a company that manufactures dishwashers. It uses aluminum to make their product. Each dishwasher requires 5 pounds of aluminum. The fixed cost is \$100 and the selling price is \$350.
(a) What is the unhedged profit one year from now?
(b) Alumco could faces severe loses if the price of the aluminum goes very high. To hedge risk, the company buys forward contract with the current forward price of \$40 per pound. What is the profit one year from now per dishwasher?
(c) Another alternative to hedge the aluminum price risk is to buy a call. Suppose that Alumco buys a 35-strike call with an expiration date one year from now, nominal amount 5 lbs, and for a premium of \$7.609104. The one year continuously compounded interest rate is 6%. Find the profit one year from now per dishwasher.
(d) Another alternative to hedge the aluminum price risk is to sell a put. Suppose that Alumco sells a 35-strike put with an expiration date one year from now, nominal amount 5 lbs, and for a premium of \$7.609104. Find the profit one year from now per dishwasher.

Solution.
(a) Let P_1 be the price of one pound of aluminum one year from now. The profit from selling one dishwasher one year from now is given by

$$350 - (5P_1 + 100) = 250 - 5P_1.$$

The unisured profit is shown in Figure 75.4.
(b) The profit of entering a long forward is

$$350 - (5 \times 40 + 100) = \$50.$$

The graph of this profit is shown in Figure 75.4.
(c) The profit that results from buying the call is

$$250 - 5S_1 + 5\max\{S_1 - 35, 0\} - 5(7.609104)e^{0.06} = 209.602 - 5S_1 + 5\max\{S_1 - 35, 0\}.$$

The graph of the profit is shown in Figure 75.4.
(d) The profit that results from selling the put is

$$250 - 5S_1 - 5\max\{35 - S_1, 0\} + 5(7.609104)e^{0.06} = 290.40 - 5S_1 - 5\max\{35 - S_1, 0\} \blacksquare$$

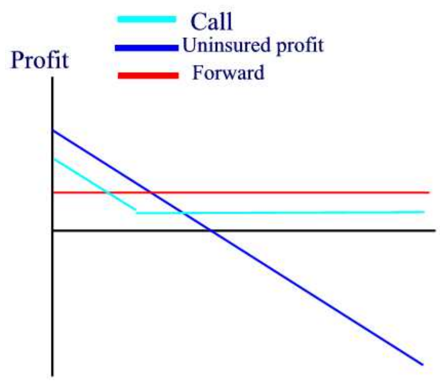


Figure 75.4

Why do firms manage risks through derivatives?
Firms engage in hedging for a variety of reasons namely:
• **To lower expected taxes**
• **To lower financial distress costs:** A large loss can threaten the survival of a firm. For example, a money-losing firm may be unable to meet fixed obligations (such as, debt payments and wages). If a firm appears in financial distress, customers may be less willing to purchase its goods (for the fear that the firm may go out of business and therefore will not provide warranty for its goods). A firm uses derivatives to transfer income from profit states to loss states. Hedging allows a firm to reduce the probability of bankruptcy or financial distress.
• **To lower costly external finance:** Usually when a form is in a loss state they have to offset that by either using cash reserve or raising funds (by borrowing or issuing bonds). Borrowing entails costs such as underwriting fees and high interest rates (because the lender fears for his money when the borrower is in s loss state). If a firm uses its cash reserve to pay for the losses, this reduction in cash can increase the probability of costly external financing and can lead a firm to forego investment projects it would have taken had cash been available to use for financing. Thus, hedging with derivatives can safeguard cash reserves and reduce the probability of raising funds externally.
• **Increasing debt capacity:** That is the amount a company can borrow.
• **Managerial risk aversion:** Firm managers are typically not well-diversified. Salary, bonus,

and compensation are tied to the performance of the firm. Poor diversification makes managers risk-averse, i.e., they are harmed by a dollar of loss more than they are helped by a dollar of gain. Managers have incentives to reduce uncertainty through hedging.

As there are reasons for hedging, there are also reasons why a company may choose not to hedge, namely:

- Transaction costs of engaging in hedges (such as commissions and the bid-ask spread)
- The cost of expertise required to analyze a hedging strategy
- The cost of monitoring and controlling the hedging transactions.
- Potential collateral requirements associated with some types of hedging.
- The tax and accounting consequences of hedges.

In the real world, small companies are discouraged to do derivatives because the reasons above. However, large companies have financial, accounting and legal departments which allow them to take advantage of the opportunities on market derivatives. The financial department of a large company can asses derivatives as well or better than the market does. Their legal and accounting departments allow them to take advantage of the current tax laws.

Answer Key

The answer key to the book can be requested directly from the author through email: mfinan@atu.edu

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